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► **To cite this version:**

Laure Coutin, Laurent Decreusefond. Invertibility of functionals of the Poisson process and applications. 2023. hal-04153437

**HAL Id: hal-04153437**

**<https://telecom-paris.hal.science/hal-04153437>**

Preprint submitted on 6 Jul 2023

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# Invertibility of functionals of the Poisson process and applications

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2023

We show that solving SDEs with constant volatility on the Wiener space is the analog of constructing Hawkes-like processes, i.e. self excited point process, on the Poisson space. Actually, both problems are linked to the invertibility of some transformations of the sample paths which respect absolute continuity: adding an adapted drift for the Wiener space, making a random time change for the Poisson space. Following previous investigations by Üstünel on the Wiener space, we establish an entropic criterion on the Poisson space which ensures the invertibility of such a transformation. As a consequence of this criterion, we improve the variational representation of the entropy with respect to the Poisson process distribution. Pursuing the Wiener-Poisson analogy so established, we define several notions of generalized Hawkes processes as weak or strong solutions of some fixed point equations and show a Yamada-Watanabe like theorem for these new equations. As a consequence, we find another construction of the classical (even non linear) Hawkes processes without the recourse to a Poisson measure.

Keywords: Girsanov Theorem, Hawkes processes, Invertibility, Random Time Change

Math Subject Classification: 60G55

## 1 Introduction

The simplest non linear stochastic differential equations are those of the form:

$$X(0) = 0 \text{ and } dX(t) = \dot{u}(X(t)) dt + dB(t) \iff X(t) = \int_0^t \dot{u}(X(s)) ds + B(t), \quad (1)$$

where  $B$  is a standard Brownian motion. Denote by  $W$  the space of continuous functions over  $[0, 1]$ , null at time 0, equipped with the Wiener measure  $\mu$ . We may view the process  $X$  as a map from  $W$  into itself by the construction:

$$\begin{aligned} X : W &\longrightarrow W \\ \omega &\longmapsto \left( t \mapsto X(\omega, t) \right). \end{aligned}$$

Consider also the map:

$$\begin{aligned} U : W &\longrightarrow W \\ \omega &\longmapsto \left( t \mapsto \omega(t) - \int_0^t \dot{u}(\omega(s)) \, ds \right). \end{aligned} \quad (2)$$

Then, (1) is equivalent to say that  $X$  is a solution of the equation

$$U \circ X = \text{Id}_W.$$

Hence, to solve (1) is to invert  $U$ . In [26] and further on in [15, 20, 27], the authors showed that  $U$  is invertible if and only if

$$H(U^\# \mu | \mu) = \frac{1}{2} \mathbf{E} \left[ \int_0^1 \dot{u}(s)^2 \, ds \right] \quad (3)$$

where  $H(U^\# \mu | \mu)$  is the relative entropy between  $U^\# \mu$ , the image measure of  $\mu$  by  $U$ , and  $\mu$  itself. The key ingredient of the proofs is the quasi-invariance (or Girsanov) theorem. Since we also have a Girsanov theorem on the Poisson space (the space of configurations equipped with  $\pi$ , the probability that makes the canonical process a Poisson process of intensity 1), it is natural to ponder on the analogue of the notion of invertibility on this space. It began as an intuition and turned into a deep result that this notion is intimately linked not to solutions of Poisson driven stochastic differential equations but to the construction of Hawkes processes.

To grasp the difference between the Wiener and Poisson settings, we must go back to the basics of the Girsanov theorem. A measure  $\nu$  on  $W$  is absolutely continuous with respect to  $\mu$  if there exists an adapted process  $u$  in the Cameron-Martin space such  $B - u$  is a  $\nu$ -local martingale of square bracket  $(t \mapsto t)$ . Since the square bracket of a semi-martingale is unchanged by the addition of a finite variation process, the Lévy's characterization theorem says that  $B - u$  is a  $\nu$  Brownian motion. This is this quasi-invariance which is used to prove that SDE's like (1), have weak solutions under mild assumptions on  $\dot{u}$  and which is the key to the investigations about invertibility.

Now, let  $\mathfrak{N}$  be the space of configurations on  $(0, +\infty)$  (see the exact definition below) and  $\pi$  the probability on  $\mathfrak{N}$  such that the canonical process  $N$  is a Poisson process of unit intensity. The Girsanov theorem says that if  $\nu$  is absolutely continuous with respect to  $\pi$ , there exists a non-decreasing predictable process, denoted by  $y$ , such that  $N - y$  is a  $\nu$  local martingale. The difference here is that the  $\nu$ -compensator of  $N$  is  $y$  so neither  $N$  nor  $N - y$  are Poisson processes under  $\nu$ . We have thus to construct another transformation of the sample-paths of  $N$  such that the process we obtain is a  $\nu$ -Poisson process of unit intensity. This question has been seldom addressed (see [4, 8]) and not in a form which is convenient for our present goal. It turns out that it is the process  $N(y^*(t))$ , where  $y^*$  is the right-inverse of  $y$ , which plays the role of  $B - u$  in the Poisson space. Thus, the analog of the map  $U$  defined in (2) is the map  $\mathbf{Y}$

$$\begin{aligned} \mathbf{Y} : \mathfrak{N} &\longrightarrow \mathfrak{N} \\ N &\longmapsto N \circ y^*. \end{aligned}$$

If this map has a reciprocal map  $\mathbf{Z}$  associated to a process  $z$ , then the process  $Z$  has  $z^*$  as a compensator, which turns to be equal to  $y(Z, \cdot)$ . Thus  $Z$  is a generalized Hawkes process (g-Hawkes for short), i.e. a process whose compensator depends on itself (see Theorem 4.4 for a

precise statement). This means that constructing a g-Hawkes process is the Poissonian analogue to solving SDEs like (1) in the Wiener space. This point of view, which, to the best of our knowledge, is new, has major consequences as we can now transfer the problems known on SDEs (weak, strong and martingales solutions, perturbations, stationarity, etc.) to g-Hawkes processes. We only focus here on the different notions of solutions for the g-Hawkes problem and show the analogue of the Yamada-Watanabe theorem. We can then construct a classical, possibly with a non linear intensity, Hawkes process without the recourse to a larger probability space as usually done ([11, 22, 16] and references therein).

The variational representation of entropy is a crucial theorem of the theory of large deviations [14], also known as the Boué-Dupuis or Borell formula [5, 6] for the Gaussian measure on  $\mathbf{R}^n$ . It has been extended to the Wiener space by Léhec in [21] and with weaker hypothesis by Üstünel in [26], as a consequence of his entropic criterion for the invertibility of  $U$ . Actually, the core of the proof is to invert a map like  $U$ . We follow here the same lines to establish an analog formula for the Poisson space. A similar result already exists in [28] but require to embed the Poisson process in the wider probability space of marked point processes.

This paper is organized as follows: In Section 2, we define what we consider here as random time changes and describe their action on stochastic integrals. In Section 3, we establish the quasi-invariance theorem and compute the Radon-Nikodym density of the push-forward of the measure  $\pi$  by a random time change. In Section 4, we combine these results to obtain the entropic criterion which guarantees the invertibility of a map like  $Y$ . We take profit of the robustness of the entropic criterion with respect to limit procedure to obtain a variational representation of the entropy in Section 5. The notion of weak, strong and martingale generalized Hawkes problem are introduced in Section 6. We then prove the Yamada-Watanabe theorem for this new kind of equations.

## 2 Transformations of measures on the configuration space

**Definition 2.1** Let  $\mathfrak{N}$  be the set of locally finite, simple configurations on  $E = (0, +\infty)$  equipped with the vague convergence. We denote by  $\omega$  its generic element. For each  $\omega \in \mathfrak{N}$ , there exists  $(T_n(\omega), n \geq 1)$  such that  $T_n(\omega) < T_{n+1}(\omega)$ ,  $T_n(\omega)$  tends to infinity as  $n$  tends to infinity and

$$\omega = \sum_{n \geq 1} \epsilon_{T_n(\omega)}$$

where  $\epsilon_a$  is the Dirac mass at  $a$ . We denote by  $N$ , the counting process associated to this measure by:

$$N(\omega, t) = \omega([0, t]).$$

Note that given the sample-path of  $N$ , we can retrieve  $\omega$  as

$$T_n(\omega) = \inf\{t, N(t) = n\} \text{ or } (T_n(\omega), n \geq 1) = (t, \Delta N(t) = 1)$$

where  $\Delta N(t) = N(t) - N(t^-)$ . This means that we can identify a sample-path of  $N$  with an element of  $\mathfrak{N}$ . A point process is a random variable with values in  $\mathfrak{N}$ .

The filtrations do play an important role in the following. We denote by  $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$  a generic right continuous filtration on  $\mathfrak{N}$ . We denote by  $\mathcal{F}_\infty$  the whole  $\sigma$ -field, i.e.  $\mathcal{F}_\infty =$

$\bigvee_{t \geq 0} \mathcal{F}_t$ . The minimal filtration under which the canonical process  $N$  is measurable is

$$\mathcal{N}_t = \sigma\{N(s), s \leq t\}. \quad (4)$$

We follow the presentation of [18] where the notion of predictability is defined without any reference to the completion of the filtration.

**Definition 2.2** For any filtration  $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$  on  $\mathfrak{N}$ , a real-valued process  $(X(t), t \geq 0)$  is called  $\mathcal{F}$ -predictable and belongs to  $\mathcal{P}(\mathcal{F})$ , if the application  $(\omega, t) \mapsto X(\omega, t)$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{P}$  on  $\mathfrak{N} \times [0, +\infty)$  generated by the applications  $(\omega, t) \mapsto Y(\omega, t)$  which are  $\mathcal{F}_t$  measurable in  $\omega$  and left-continuous in  $t$ .

■ **Remark 1** The usual definition of predictability (see [12, 13]) is similar but requires the filtration  $\mathcal{F}$  to be complete with respect to a given probability measure  $\mu$ . Let us denote by  $\mathcal{F}^\mu$  the completion of  $\mathcal{F}$  with respect  $\mu$ . According to [12], for any predictable process in our sense there always exists a predictable  $\mathcal{F}^\mu$  process such that the two processes coincide on a  $\mu$ -null set. Hence all the usual results on predictable processes are still valid with our current definition. ■

**Definition 2.3** Let  $\mu$  be a probability measure on  $(\mathfrak{N}, \mathcal{F}_\infty)$ . A process  $X$ , null at time 0, is said to be of  $\mu$ -finite variation if there exist  $X^+$  and  $X^-$  two right continuous, null at time 0, and non decreasing processes such that

$$X = X^+ - X^-, \mu - \text{a.s.}$$

The process  $X$  has locally integrable variation if for any  $t \geq 0$

$$\mathbf{E}_\mu [X^+(t) + X^-(t)] < \infty.$$

Its  $\mathcal{P}(\mathcal{F}, \mu)$ -dual predictable projection is the unique predictable process  $X^p$  such that  $\tilde{X} = X - X^p$  is a  $(\mathcal{F}, \mu)$ -local martingale. When  $X$  is a counting process,  $X^p$  is called the compensator of  $X$ .

We denote by  $\mathcal{P}(\mathcal{F})$  the set of processes which are  $\mathcal{F}$ -predictable.

**Theorem 2.1** Let  $\mathcal{F}$  be a filtration on  $\mathfrak{N}$  and  $\mu$  a probability measure on  $(\mathfrak{N}, \mathcal{F}_\infty)$ .

1. Then, there exists a unique predictable (up to  $\mu$ -null set) process  $y$  which is non-decreasing, right continuous, null at time 0 and such that for any  $q \geq 1$ ,

$$t \mapsto N(t \wedge T_q(N)) - y(t \wedge T_q(N))$$

is a uniformly integrable  $(\mathfrak{N}, \mathcal{F}, \mu)$ -martingale.

2. For the converse, we need to assume that the filtration is the minimum filtration under which the canonical process is measurable. For any process  $y \in \mathcal{P}(\mathcal{N})$ , non-decreasing, null at time 0 and right-continuous, there exists a unique probability measure on  $(\mathfrak{N}, \mathcal{N}_\infty)$  such that  $N - y$  is a  $(\mathcal{N}_t, t \geq 0)$  local martingale and  $(T_q(N), q \geq 1)$  is a localizing sequence.

We denote this measure by  $\pi_y$ . When  $y = \text{Id}$ , i.e., when the canonical process is a unit rate Poisson process, we prefer to use the notation  $\pi$  instead of  $\pi_{\text{Id}}$ .

■ **Remark 2** Note that if  $y$  is deterministic then we can get rid of the hypothesis of minimal filtration as any point process with deterministic compensator is known to be a, possibly non homogeneous, Poisson process. This is known as the Watanabe characterization of the Poisson process. ■

## 2.1 Absolute continuity and equivalence

We summarize the results on local absolute continuity on the Poisson space which can be found in theorems 8.32 and 8.35 and corollary 8.37 of [19].

**Theorem 2.2** Consider a filtration  $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$  on  $\mathfrak{N}$ . Let  $\dot{\kappa}$  be a non-negative, locally integrable  $\mathcal{F}$ -predictable process. We set

$$\kappa(t) = \int_0^t \dot{\kappa}(s) \, ds.$$

We consider  $\pi_\kappa$  as defined above.

1. If a probability measure  $\nu$  on  $\mathfrak{N}$  is locally absolutely continuous with respect to  $\pi_\kappa$  along  $\mathcal{F}$  (that is denoted  $\nu \ll_{\text{loc}, \mathcal{F}} \pi_\kappa$ ) then there exists a unique process  $\dot{y}_\kappa$  which is non-negative and  $\mathcal{F}$ -predictable such that

$$\forall t \geq 0, \nu \left( \int_0^t \left( 1 - \sqrt{\dot{y}_\kappa(s)} \right)^2 \dot{\kappa}(s) \, ds < \infty \right) = 1 \quad (5)$$

and

$$t \mapsto N(t) - \int_0^t \dot{y}_\kappa(s) \dot{\kappa}(s) \, ds \text{ is a } (\mathcal{F}, \nu)\text{-local martingale.} \quad (6)$$

We set

$$y_\kappa(t) = \int_0^t \dot{y}_\kappa(s) \dot{\kappa}(s) \, ds$$

which belongs to  $\mathcal{P}(\mathcal{F})$ , is non decreasing, right continuous and null at time 0. Hence, with the notations introduced above,  $\nu = \pi_{y_\kappa}$ .

2. If  $\pi_{y_\kappa}$  is absolutely continuous with respect to  $\pi_\kappa$  on  $\mathcal{F}_\infty$  (that is denoted by  $\pi_{y_\kappa} \ll \pi_\kappa$ ) then

$$\pi_{y_\kappa} \left( \int_0^\infty \left( 1 - \sqrt{\dot{y}_\kappa(s)} \right)^2 \dot{\kappa}(s) \, ds < \infty \right) = 1. \quad (7)$$

For the converse part, we need to assume that  $\mathcal{F} = \mathcal{N}$ , the minimal filtration which makes  $N$  adapted. In this situation, consider  $\dot{y}$  a locally integrable, non negative and  $\mathcal{N}$ -predictable process and  $\pi_{y_\kappa}$  the probability measure on  $\mathfrak{N}$ , which satisfies (5) and (6).

3. Then,  $\pi_{y_\kappa}$  is locally absolutely continuous with respect to  $\pi_\kappa$  along  $\mathcal{N}$  and

$$\Lambda_{y_\kappa}(N, t) := \frac{d\pi_{y_\kappa}}{d\pi_\kappa} \Big|_{\mathcal{N}_t} = \begin{cases} \exp\left(\int_0^t \log \dot{y}_\kappa(s) dN(s) + \int_0^t (1 - \dot{y}_\kappa(s)) \dot{\kappa}(s) ds\right) & \text{if } t \leq S_m, \\ 0 & \text{if } t \geq \limsup_m S_m, \end{cases}$$

where for any integer  $m \geq 1$ ,

$$S_m = \inf \left\{ t, \int_0^t \left(1 - \sqrt{\dot{y}_\kappa(s)}\right)^2 \dot{\kappa}(s) ds \geq m \right\}.$$

4. Finally, the probability measure  $\pi_{y_\kappa}$  is absolutely continuous with respect to  $\pi_\kappa$  on  $(\mathfrak{N}, \mathcal{N}_\infty)$  if and only if (6) and (7) are satisfied.

The next result gives some necessary and sufficient conditions for the equivalence of  $\pi_y$  and  $\pi_\kappa$ , see Proposition (7.11) of [19].

**Lemma 2.3** Assume that  $\nu$  is absolutely continuous with respect to  $\mu$  on  $\mathcal{N}_\infty$  and set

$$\Lambda(t) = \frac{d\nu}{d\mu} \Big|_{\mathcal{N}_t}. \quad (8)$$

Then  $\nu$  and  $\mu$  are equivalent if and only if the following two conditions are satisfied:

i) The local martingale  $(\Lambda(t), t \geq 0)$  is uniformly integrable, i.e. there exists  $\Lambda \in L^1(\mu)$  such that

$$\Lambda(t) = \mathbf{E}_\mu[\Lambda | \mathcal{N}_t].$$

ii) The random variable  $\Lambda$  is positive  $\mu$ -a.s.

In view of this theorem, we introduce the following sets of processes.

**Definition 2.4** Let  $\mathcal{F}$  be a filtration on  $\mathfrak{N} \times \mathbf{R}^+$  and  $\dot{\kappa}$  a non-negative predictable process. Consider the probability measure  $\pi_\kappa$  on  $\mathfrak{N}$  and let  $\mathcal{P}^{++}(\mathcal{F}, \pi_\kappa)$  (respectively  $\mathcal{P}^+(\mathcal{F})$ ) be the set of positive (respectively non-negative)  $\mathcal{F}$ -predictable processes  $\dot{y}$  such that for any  $t \geq 0$

$$\pi_\kappa(y_\kappa(t) < +\infty) = 1 \text{ and } \pi_\kappa\left(\lim_{t \rightarrow \infty} y_\kappa(t) = +\infty\right) = 1$$

where the process  $y$  is defined by

$$y_\kappa(\omega, t) = \int_0^t \dot{y}(\omega, s) \dot{\kappa}(s) ds.$$

We also introduce the following subset of  $\mathcal{P}^{++}(\mathcal{F}, \pi_\kappa)$ :

$$\mathcal{P}_2^{++}(\mathcal{F}, \pi_\kappa) = \left\{ \dot{y} \in \mathcal{P}^{++}(\mathcal{F}, \pi_\kappa), \pi_\kappa \left( \int_0^\infty (1 - \sqrt{\dot{y}(s)})^2 \dot{\kappa}(s) ds < \infty \right) = 1 \right\}.$$

The most restricted class we consider is  $\mathcal{P}_\infty^{++}(\mathcal{F}, \pi_\kappa)$  of processes  $\dot{y} \in \mathcal{P}^{++}(\mathcal{F}, \pi_\kappa)$  for which there exist  $\varepsilon \in (0, 1)$  and  $T > 0$  such that

$$\varepsilon \leq \dot{y}(s) \leq \frac{1}{\varepsilon}, \quad \forall s \geq 0 \text{ and } \dot{y}(s) = 1 \text{ for } s \geq T, \quad \pi_\kappa - \text{a.s.}$$

Note that if  $\dot{y}$  belongs to  $\mathcal{P}_\infty^{++}(\mathcal{F}, \pi_\kappa)$  then  $(\Lambda_{y_\kappa}(N, t), t \geq 0)$  is uniformly integrable.

As long as there is a finite number of jumps of the process  $N$ , which happens on any finite interval, the argument of the exponential in  $\Lambda_{y_\kappa}$  cannot be minus infinity hence  $\Lambda_{y_\kappa}$  is positive. We give a sufficient condition which ensures that this may not happen even on the half line.

**Lemma 2.4** *If the predictable process  $(\dot{y} - 1)$  belongs to  $L^1(\mathfrak{N} \times \mathbf{R}^+, \pi_\kappa \otimes \dot{\kappa}(s) ds)$  then,  $\pi_\kappa$ -a.s.,*

$$\begin{aligned} \Lambda_{y_\kappa} &:= \lim_{t \rightarrow \infty} \Lambda_{y_\kappa}(t) \\ &= \exp \left( \int_0^\infty \log \dot{y}(s) dN(s) + \int_0^\infty (1 - \dot{y}(s)) \dot{\kappa}(s) ds \right) \in (0, +\infty). \end{aligned} \quad (9)$$

*Proof.* Since, for any  $x \geq -1$ ,

$$(1 - \sqrt{1+x})^2 \leq |x|,$$

we have

$$\mathbf{E}_{\pi_\kappa} \left[ \int_0^\infty (1 - \sqrt{\dot{y}(s)})^2 \dot{\kappa}(s) ds \right] \leq \mathbf{E}_{\pi_\kappa} \left[ \int_0^\infty |\dot{y}(s) - 1| \dot{\kappa}(s) ds \right] < \infty.$$

Hence, for  $\pi_\kappa$  almost all  $\omega$ , there exists  $m(\omega) < \infty$  such that  $S_m(\omega) = \infty$  for any  $m \geq m(\omega)$  and then we have

$$\begin{aligned} \Lambda_{y_\kappa} &= \lim_{t \rightarrow \infty} \Lambda_{y_\kappa}(t) \\ &= \left( \prod_{n=1}^\infty \dot{y}(T_n) \right) \exp \left( \int_0^\infty (1 - \dot{y}(s)) \dot{\kappa}(s) ds \right). \end{aligned}$$

The infinite product is convergent as soon as

$$\sum_{n=1}^\infty |\dot{y}(T_n) - 1| < \infty.$$

Since

$$\mathbf{E}_{\pi_\kappa} \left[ \sum_{n=1}^\infty |\dot{y}(T_n) - 1| \right] = \mathbf{E}_{\pi_\kappa} \left[ \int_0^\infty |\dot{y}(s) - 1| \dot{\kappa}(s) ds \right],$$

we see that  $(\dot{y} - 1) \in L^1(\mathfrak{N} \times \mathbf{R}^+, \pi_\kappa \otimes \dot{\kappa}(s) ds)$  entails (9). □



## 2.2 Random time changes

The sequel of this paper will be based on the notion of random time change. We refer to [19, chapter 10] or [3] for details on this notion and its links with stochastic calculus.

**Definition 2.5** On  $(\mathfrak{N}, \mathcal{F} = (\mathcal{F}_t, t \geq 0))$ , a random change of time is a right continuous, null at time 0, non-decreasing process  $(\eta(t), t \geq 0)$  such that

$$\eta(t) < \infty, \forall t \geq 0, \lim_{t \rightarrow \infty} \eta(t) = +\infty$$

and for any  $t \geq 0$ ,  $\eta(t)$  is an  $\mathcal{F}$ -stopping time. We denote by  $\mathcal{F}^\eta$  the filtration  $(\mathcal{F}_{\eta(t)}, t \geq 0)$ . For  $X$  a process,  $\tau_\eta(X)$  is the process defined by

$$\tau_\eta(X)(t) = X(\eta(t)).$$

Note that many change of time are given by the right inverse of non decreasing predictable processes: For  $y$  such a process,

$$y^*(\omega, t) = \inf\{s, y(\omega, s) > t\}$$

is a change of time (with the usual convention that the infimum of the empty set is infinite). According to [19, Chapter 10],

$$(y(\omega, t) < s) = (y^*(\omega, s^-) > t) \quad (10)$$

$$(y^*(\omega, t) < s) = (y(\omega, s^-) > t). \quad (11)$$

If  $y$  is continuous, we have

$$y(\omega, y^*(\omega, t)) = t \text{ and } y^*(\omega, y(\omega, t)) = \inf\{u, y^*(\omega, u) > y^*(\omega, t)\}.$$

Note that if  $y$  belongs to  $\mathcal{P}^{++}(\mathcal{F}, \mu)$ , then  $y$  is almost-surely an homeomorphism from  $\mathbf{R}^+$  onto itself, hence

$$y(\omega, y^*(\omega, t)) = y^*(\omega, y(\omega, t)) = t \text{ and } \tau_{y^*}(X)(t) = X(y^*(t)), \text{ for all } t \geq 0, \mu\text{-a.s.} \quad (12)$$

Furthermore, (10) entails that  $y(\omega, t)$  is an  $\mathcal{F}^{y^*}$  stopping time.

**Definition 2.6** Consider  $m$ , the smooth, convex, non-negative function defined on  $[-1, \infty)$  by

$$m(x) = \begin{cases} (x+1) \log(x+1) - x & \text{if } x > -1, \\ 1 & \text{if } x = -1. \end{cases}$$

and  $L_m$ , the corresponding Orlicz space [1]:

$$L_m = \left\{ f : \mathbf{R}^+ \rightarrow [-1, \infty), \int_0^\infty m(f(s)) ds < \infty \right\}.$$

We view the time change as a map from  $\mathfrak{N}$  into itself.

**Definition 2.7** For  $\dot{y} \in \mathcal{D}^{++}(\mathcal{N})$ , let

$$\begin{aligned} \mathbf{Y} &: \mathfrak{N} \longrightarrow \mathfrak{N} \\ N &\longmapsto \tau_{y^*}(N) : \left( t \mapsto N(y^*(N, t)) \right). \end{aligned} \quad (13)$$

We denote by  $Y$  the process  $\tau_{y^*}(N)$ , also equal to  $\mathbf{Y}(N)$ .

**Definition 2.8** For a probability measure  $\mu$  on  $\mathfrak{N}$ , we denote by  $\mathbf{Y}^\# \mu$  the distribution of the process  $Y$  on  $\mathfrak{N}$  or equivalently the push-forward of the measure  $\mu$  by the map  $\mathbf{Y}$ .

One way to understand this transformation is to consider that the locations of the atoms of the underlying configuration  $N$  are fixed once and for all. They are discovered at unit rate with  $N$  and at random speed by  $Y$ : The  $k$ -th jump of  $Y$  occurs at time  $t$  if and only if  $y^*(N, t) = T_k(N)$  hence

$$y^*(N, T_k(Y)) = T_k(N). \quad (14)$$

■ **Remark 3** Note that the process  $Y$  is adapted to the filtration

$$\mathcal{F}^{y^*} = (\mathcal{F}_{y^*(t)}, t \geq 0)$$

and has  $(\mathcal{F}^{y^*}, \pi)$ -compensator  $y^*$  (see [19, Theorem 10.17]).

**A word of caution:** The process  $y(t)$  is predictable hence depends on a source of randomness. Along the text, we may have to change this source, i.e. we may have to consider  $y(N, t)$  or  $y(Z, t)$  where  $Z$  is another point process. For the sake of notations, when we use  $y(t)$ , it must be understood as  $y(N, t)$  where  $N$  is the canonical point process. When we evaluate  $y$  along the sample-path of another point process,  $q$  as in  $y(Z, t)$ , it will be indicated.

**Lemma 2.5** For  $\dot{y} \in \mathcal{D}^{++}(\mathcal{N})$ , let  $\mathcal{Y}_t = \sigma(Y(u), u \leq t)$ . Then, we have

$$\mathcal{Y}_t \vee \sigma(y^*(s), s \leq t) = \mathcal{N}_{y^*(t)}.$$

Furthermore,  $\mathcal{Y}_\infty \subset \mathcal{N}_\infty^{y^*} = \mathcal{N}_\infty$ .

*Proof.* For any  $s \leq t$ ,

$$y^*(s) \in \mathcal{N}_{y^*(s)} \subset \mathcal{N}_{y^*(t)}.$$

The definition of  $Y$  induces that  $Y(t)$  is  $\mathcal{N}_{y^*(t)}$  measurable hence

$$\mathcal{Y}_t \subset \mathcal{N}_{y^*(t)}.$$

It follows that

$$\mathcal{Y}_t \vee \sigma(y^*(s), s \leq t) \subset \mathcal{N}_{y^*(t)}.$$

Conversely, let  $A \in \mathcal{N}_{y^*(t)}$ , according to [19, Proposition 3.40],

$$\mathbf{1}_A = \sum_{q=0}^{\infty} \Psi_q \left( T_1(N), \dots, T_q(N) \right) \mathbf{1}_{\{T_q(N) \leq y^*(N, t) < T_{q+1}(N)\}},$$

where  $\Psi_q$  is measurable from  $(\mathbf{R}^+)^q$  to  $\{0, 1\}$ . It follows from the definition of  $Y$  (see (14)) that

$$\mathbf{1}_A = \sum_{q=0}^{\infty} \Psi_q \left( y^*(N, T_k(Y)), k = 1, \dots, q \right) \mathbf{1}_{\{Y(t)=q\}}.$$

Since  $y^*$  is continuous, for  $u \leq t$ , we have

$$y^*(N, u) = \lim_{n \rightarrow \infty} \sum_{i=0}^{2^n-1} y^*(N, \frac{i}{2^n}) \mathbf{1}_{[\frac{i-1}{2^n}, \frac{i}{2^n})}(u),$$

thus, for  $k \leq q$ ,

$$y^*(N, T_k(Y)) \mathbf{1}_{\{Y(t)=q\}} \in \mathcal{Y}_t \wedge \sigma(y^*(s), s \leq t).$$

Hence the converse embedding also holds.

The inclusion  $\mathcal{Y}_\infty \subset \mathcal{N}_\infty^{y^*}$  is immediate from the first part of the proof. The equality between  $\mathcal{N}_\infty^{y^*}$  and  $\mathcal{N}_\infty$  comes from [19, page 326].  $\square$

The following theorem is Theorem (10.19) of [19] and Lemma 1.3 of [3].

**Theorem 2.6** *Let  $\dot{y} \in \mathcal{P}^{++}(\mathcal{F})$ . For  $r \in \mathcal{P}(\mathcal{F})$  such that its stochastic integral is well defined, we have*

$$\tau_{y^*} \left( \int_0^\cdot r(N, s) ds \right) = \int_0^\cdot r(N, y^*(N, s)) dy^*(s) \quad (15)$$

and for the stochastic integrals,

$$\tau_{y^*} \left( \int_0^\cdot r(N, s) dN(s) \right) = \int_0^\cdot r(N, y^*(N, s)) dY(s). \quad (16)$$

We can extend  $\mathbf{Y}$  to stochastic processes which are measurable maps from  $(\mathfrak{N} \times \mathbf{R}^+, \mathcal{N}_\infty \otimes \mathcal{B}(\mathbf{R}^+))$  to  $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ . For  $\mu$  a probability measure on  $\mathfrak{N}$ , we denote by  $L^0(\mathfrak{N} \times \mathbf{R}^+, \mathbf{R}; \mu)$  the set of stochastic processes equipped with the topology of convergence in probability. For the sake of simplicity, we still denote by  $\mathbf{Y}$  this extension, which is not to be confused with the process  $Y$ .

**Definition 2.9** The map  $\mathbf{Y}$  is defined by

$$\begin{aligned} \mathbf{Y} : L^0(\mathfrak{N} \times \mathbf{R}^+, \mathbf{R}; \mu) &\longrightarrow L^0(\mathfrak{N} \times \mathbf{R}^+, \mathbf{R}; \mu) \\ U &\longmapsto \left( (N, s) \mapsto U(Y, y^*(N, s)) \right). \end{aligned}$$

For instance, if we denote by  $N^a$  the process  $N$  stopped at time  $a$ :

$$N^a(t) = N(t \wedge a),$$

we have

$$\begin{aligned} Y^{y(a)}(t) &= Y(y(a) \wedge t) \\ &= N(y^*(N, y(a) \wedge t)) \\ &= N(a \wedge y^*(N, t)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} y^*(N^a)(t) &= N^a(y^*(N, t)) \\ &= N(y^*(N, t) \wedge a) \end{aligned}$$

so that

$$N^a \circ \mathbf{Y} = Y^{y(a)}.$$

We then have the following composition rules.

**Theorem 2.7** *Let  $\dot{y} \in \mathcal{P}^{++}(\mathcal{N})$ . For  $r \in \mathcal{P}(\mathcal{N})$  such that its stochastic integral with respect to  $Y$  is well defined, we have*

$$\left( \int_0^\cdot r(N, s) ds \right) \circ \mathbf{Y} = \int_0^\cdot r(Y, y^*(N, s)) dy^*(N, s) \quad (17)$$

and

$$\left( \int_0^\cdot r(N, s) dN(s) \right) \circ \mathbf{Y} = \int_0^\cdot r(Y, y^*(N, s)) dY(s). \quad (18)$$

■ **Remark 4** These formula do not hold if  $\dot{y}$  is not supposed to be positive or if  $y$  is not an homeomorphism almost surely. ■

*Proof.* Since  $y$  defines a diffeomorphism from  $\mathbf{R}^+$  onto itself, the change of variable  $u = y(N, s)$  yields

$$\begin{aligned} \left( \int_0^\cdot r(N, s) ds \right) \circ \mathbf{Y}(t) &= \int_0^{y^*(N, t)} r(Y, s) ds \\ &= \int_0^t r(Y, y^*(N, u)) \dot{y}^*(N, u) du \end{aligned}$$

and (17) holds.

To prove (18), it is sufficient to prove it for simple predictable process, i.e. we assume that

$$r(N, s) = A(N) \mathbf{1}_{(a, b]}(s)$$

for some  $A \in \mathcal{N}_a$ . We have

$$\int_0^\cdot r(N, s) dN(s) = A(N)(N^b - N^a)$$

where  $N^a$  is the process  $N$  stopped at time  $a$ . On the one hand, we have

$$\begin{aligned} \left( A(N)(N^b - N^a) \right) \circ \mathbf{Y} &= A(Y)(y^*(N^b) - y^*(N^a)) \\ &= A(Y)\left( Y^{y(b)} - Y^{y(a)} \right). \end{aligned}$$

On the other hand,

$$r(Y, y^*(s)) = A(Y) \mathbf{1}_{(y(a), y(b)]}(s)$$

hence

$$\int_0^\cdot r(Y, y^*(s)) dY(s) = A(Y) \left( Y^{y(b)} - Y^{y(a)} \right).$$

The proof is thus complete.  $\square$

### 3 Quasi-invariance

The next result is the exact analog of the quasi-invariance theorem for the Wiener space (often quoted as the Girsanov theorem): find a perturbation of the sample paths and a change of probability which compensate each other. As mentioned above, for point processes, translations by an element of the Cameron-Martin space are replaced by time changes (see [8] and [4] for marked point processes). We first evaluate how a time change modifies the Radon-Nikodym derivative.

**Lemma 3.1** *Let  $\kappa$  be a deterministic positive function from  $\mathbf{R}^+$  into itself such that*

$$\kappa(t) = \int_0^t \dot{\kappa}(s) ds < \infty, \forall t \geq 0 \text{ and } \lim_{t \rightarrow \infty} \kappa(t) = +\infty.$$

*Let  $\nu$  be locally absolutely continuous with respect to  $\pi_\kappa$  along  $\mathcal{F}$  and let  $\dot{y}_\kappa$  denote its Girsanov factor, i.e.  $\nu = \pi_{y_\kappa}$ . Assume that  $\dot{y}_\kappa$  belongs to  $\mathcal{P}_2^{++}(\mathcal{F}, \pi_{y_\kappa})$ . Then,  $\pi_{y_\kappa}$  is locally absolutely continuous with respect to  $\pi_\kappa$  along  $\mathcal{F}^{y_\kappa^*}$  and we have*

$$\Lambda_{y_\kappa^*}^*(t) := \frac{d\nu}{d\pi} \Big|_{\mathcal{F}_t^{y_\kappa^*}} = \exp \left( \int_0^t \log \left( \frac{1}{(\kappa \circ y_\kappa^*)'(s)} \right) dY(s) + \int_0^t \left( (\kappa \circ y_\kappa^*)'(s) - 1 \right) ds \right). \quad (19)$$

*Proof.* Remark that under the hypothesis  $\dot{y}_\kappa \in \mathcal{P}_2^{++}(\mathcal{F}, \pi_{y_\kappa})$ , for almost all sample paths,  $S_m$  is infinite after a certain rank thus for any  $t \geq 0$ ,

$$\Lambda_{y_\kappa}(t) = \exp \left( \int_0^t \log \dot{y}_\kappa(s) dN(s) + \int_0^t (1 - \dot{y}_\kappa(s)) \dot{\kappa}(s) ds \right). \quad (20)$$

For  $A \in \mathcal{F}_{y_\kappa^*}(t)$ , by monotone convergence, we have

$$\begin{aligned} \mathbf{E}_{\pi_{y_\kappa}} [\mathbf{1}_A] &= \lim_{s \rightarrow \infty} \mathbf{E}_{\pi_{y_\kappa}} [\mathbf{1}_A \mathbf{1}_{\{y_\kappa^*(t) \leq s\}}] \\ &= \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}_{\pi_\kappa} [\mathbf{1}_A \mathbf{1}_{\{s < S_n\}} \mathbf{1}_{\{y_\kappa^*(t) \leq s\}} \Lambda_{y_\kappa}] \\ &= \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}_{\pi_\kappa} [\mathbf{1}_A \mathbf{1}_{\{s < S_n\}} \mathbf{1}_{\{y_\kappa^*(t) \leq s\}} \mathbf{E}_{\pi_\kappa} [\Lambda_{y_\kappa}(s) \mid \mathcal{F}_{y_\kappa^*}(t) \wedge s \wedge S_n]] \\ &= \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}_{\pi_\kappa} [\mathbf{1}_A \mathbf{1}_{\{s < S_n\}} \mathbf{1}_{\{y_\kappa^*(t) \leq s\}} \Lambda_{y_\kappa}(y_\kappa^*(t) \wedge s \wedge S_n)] \end{aligned}$$

according to the stopping time theorem applied to the bounded stopping time  $y_\kappa^*(t) \wedge s$  and to the martingale  $\Lambda_{y_\kappa}^{S_n}$ . Thus, in view of (20), we obtain

$$\mathbf{E}_{\pi_y} [\mathbf{1}_A] = \mathbf{E}_\pi \left[ \mathbf{1}_A \Lambda_y \left( y_\kappa^*(t) \right) \right].$$

Then, Theorem 2.6 yields

$$\Lambda_{y_\kappa} \left( y_\kappa^*(t) \right) = \exp \left( \int_0^t \log \dot{y} \left( y_\kappa^*(s) \right) dY(s) + \int_0^t \left( 1 - \dot{y} \left( y_\kappa^*(s) \right) \right) \dot{\kappa} \left( y_\kappa^*(s) \right) y_\kappa^*(s) ds \right).$$

On the one hand, we have  $y_\kappa' = y \dot{\kappa}$  and on the other hand,

$$y_\kappa \left( y_\kappa^*(s) \right) = s \implies y_\kappa' \left( y_\kappa^*(s) \right) \left( y_\kappa^* \right)'(s) = 1.$$

It follows that

$$\dot{y} \left( y_\kappa^*(s) \right) \dot{\kappa} \left( y_\kappa^*(s) \right) \left( y_\kappa^* \right)'(s) = 1$$

and that

$$\dot{y} \left( y_\kappa^*(s) \right) = \frac{1}{\left( \kappa \circ y_\kappa^* \right)'(s)}.$$

Thus, (19) holds. □

**Theorem 3.2 — Quasi-invariance.** *Assume that the hypothesis of Lemma 3.1 hold. Then, the distribution of the process  $Y = \tau_{y_\kappa}(N)$  under  $\pi_{y_\kappa}$  is  $\pi$ . This means that for any bounded measurable  $f : (\mathfrak{N}, \mathcal{N}_\infty) \rightarrow \mathbf{R}$ , for any  $t \geq 0$ ,*

$$\mathbf{E}_{\pi_\kappa} \left[ f \left( Y^t \right) \Lambda_{y_\kappa^*}^*(t) \right] = \mathbf{E}_{\pi_{\text{Id}}} [f], \quad (21)$$

where  $Y^t$  is the process  $Y$  stopped at time  $t$ .

*Proof.* Note that by the stopping time theorem and Corollary 10.12 of [19]

$$\mathbf{E}_{\pi_\kappa} \left[ Y(t) - \kappa \left( y_\kappa^*(t) \right) \mid \mathcal{N}_{y_\kappa^*(s)} \right] = Y(s) - \kappa \left( y_\kappa^*(s) \right),$$

hence the compensator of  $Y$  under  $\pi_\kappa$  is  $\kappa \circ y_\kappa^*$ . Thus,

$$\begin{aligned} R(t) &= \int_0^t \frac{1}{\left( \kappa \circ y_\kappa^* \right)'(s)} dY(s) - \int_0^t \frac{1}{\left( \kappa \circ y_\kappa^* \right)'(s)} \left( \kappa \circ y_\kappa^* \right)'(s) ds \\ &= \int_0^t \frac{1}{\left( \kappa \circ y_\kappa^* \right)'(s)} dY(s) - t \end{aligned}$$

is a  $(\mathcal{F}^{y_\kappa^*}, \pi_\kappa)$  local martingale. The standard Girsanov theorem [19, Theorem 7.24]

$$R(t) - \int_0^t \frac{1}{\Lambda_{y_\kappa^*}^*(s)} d[R, \Lambda_{y_\kappa^*}^*](s)$$

is a  $(\mathcal{F}^{Y_\kappa^*}, \pi_{Y_\kappa^*})$  local martingale. Note that  $R$  and  $\Lambda_{Y_\kappa^*}^*$  have the same jump times as  $Y$ , hence

$$\begin{aligned} \int_0^t \frac{1}{\Lambda_{Y_\kappa^*}^*(s)} d[R, \Lambda_{Y_\kappa^*}^*](s) &= \sum_{s \leq t, \Delta Y(s) \neq 0} \frac{1}{\Lambda_{Y_\kappa^*}^*(s)} \Delta R(s) \Delta \Lambda_{Y_\kappa^*}^*(s) \\ &= \sum_{s \leq t, \Delta Y(s) \neq 0} \left(1 - \frac{\Lambda_{Y_\kappa^*}^*(s^-)}{\Lambda_{Y_\kappa^*}^*(s)}\right) \Delta R(s) \\ &= \sum_{s \leq t, \Delta Y(s) \neq 0} \left(1 - (\kappa \circ Y_\kappa^*)'(s)\right) \frac{1}{(\kappa \circ Y_\kappa^*)'(s)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} R(t) - \int_0^t \frac{1}{\Lambda_{Y_\kappa^*}^*(s)} d[R, \Lambda_{Y_\kappa^*}^*](s) &= \int_0^t \left[ \frac{1}{(\kappa \circ Y_\kappa^*)'(s)} - \left( \frac{1}{(\kappa \circ Y_\kappa^*)'(s)} - 1 \right) \right] dY(s) - t \\ &= Y(t) - t. \end{aligned}$$

This means that  $Y$  has  $(\mathcal{F}^{Y_\kappa^*}, \pi_{Y_\kappa^*})$ -compensator  $(t \mapsto t)$ . According to Theorem 2.1,  $Y$  is an  $\mathcal{F}^{Y_\kappa^*}$ -adapted unit Poisson process under  $\pi_{Y_\kappa^*}$ .  $\square$

■ **Remark 5** The process  $Y_\kappa = (Y(\kappa(t)), t \geq 0)$  is then adapted to the filtration  $\mathcal{G} = (\mathcal{N}_{\kappa(t)}^{Y_\kappa^*}, t \geq 0)$  and is a  $\pi_\kappa$  point process of compensator  $\kappa$  and then we have

$$\mathbf{E}_{\pi_\kappa} \left[ f(Y_\kappa^t) \Lambda_{Y_\kappa^*}^*(\kappa(t)) \right] = \mathbf{E}_{\pi_\kappa} [f].$$

This invariance formula is the key to our investigations. In full generality, we could continue in such a general setting with any  $\dot{\kappa}$ . Actually, changing  $\dot{\kappa}$  amounts to change the clock with which time is measured. As long as this clock is deterministic and increasing, this does not change the essence of the results to come so there is no loss in generality but a great gain in simplicity and clarity to focus on the situation where  $\kappa = \text{Id}$ . For the sake of notations, we suppress the subscript  $\kappa$  hereafter.  $\blacksquare$

The following theorems are often written in terms of  $\dot{y}^*$ , this can be translated in terms of  $\dot{y}$  as shows the next lemma.

**Lemma 3.3** For  $\dot{y} \in \mathcal{P}_2^{++}(\mathcal{N})$ ,

$$\begin{aligned} \Lambda_y &= \exp \left( \int_0^\infty \log(\dot{y}(s)) dN(s) + \int_0^\infty (1 - \dot{y}(s)) ds \right) \\ &= \Lambda_{y^*} \end{aligned} \tag{22}$$

$$= \exp \left( - \int_0^\infty \log(\dot{y}^*(s)) dY(s) + \int_0^\infty (\dot{y}^*(s) - 1) ds \right) \tag{23}$$

Furthermore, for any  $p \geq 1$ ,

$$\int_0^\infty (\dot{y}^*(s) \log(\dot{y}^*(s)) - \dot{y}^*(s) + 1) ds = \int_0^\infty (\dot{y}(s) - 1 - \log(\dot{y}(s))) ds \quad (24)$$

$$\int_0^\infty \left| \frac{1}{\dot{y}^*(s)} - 1 \right|^p \dot{y}^*(s) ds = \int_0^\infty |\dot{y}(s) - 1|^p ds. \quad (25)$$

*Proof.* According to Theorem 2.2 for  $\dot{y} \in \mathcal{D}_2^+(\mathcal{N})$ , we have

$$\Lambda_y = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_\infty} \quad \text{and} \quad \Lambda_{y^*} = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_\infty^{y^*}},$$

(22) comes from Lemma 2.5. Eqn. (23) is a straightforward consequence of (19). Identities (24) and (25), since the integrands are non negative, are obtained through the change of variable  $s = y(u)$  and the relation

$$\dot{y}^*(y(u)) = \frac{1}{\dot{y}(u)}.$$

□

We now prove that well behaved time changes induce locally absolutely continuous probability on  $\mathfrak{N}$ .

**Theorem 3.4** *Let  $\dot{y}$  belong to  $\mathcal{D}^{++}(\mathcal{N})$  such that  $\pi_y$  is equivalent to  $\pi$  on  $\mathcal{N}_\infty$ . Then  $\mathbf{Y}^\# \pi$  is equivalent to  $\pi$  on  $\mathcal{N}_\infty$ .*

*Proof.* Since  $\pi_y$  is equivalent to  $\pi$  on  $\mathcal{N}_\infty$ ,  $\mathbf{Y}^\# \pi_y$  is equivalent to  $\mathbf{Y}^\# \pi$ . Furthermore, we have

$$\Lambda_{y^*(t)} = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_{y^*(t)}},$$

hence,

$$\lim_{t \rightarrow \infty} \Lambda_{y^*(t)} = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_\infty} = \Lambda_y.$$

According to Lemma 2.3, the martingale

$$\left( \Lambda_{y^*(t)}, t \geq 0 \right)$$

is then uniformly integrable and we can let  $t$  go to infinity in (21) to obtain

$$\mathbf{Y}^\# \pi_y = \pi.$$

As a consequence,

$$\pi = \mathbf{Y}^\# \pi_y \sim \mathbf{Y}^\# \pi.$$

The proof is thus complete.

□



## 4 Invertibility

**Definition 4.1** For a probability  $(\mu, \mathcal{F}_\infty)$  on  $\mathfrak{N}$ , a map  $\mathfrak{Y} : \mathfrak{N} \rightarrow \mathfrak{N}$  is left invertible if and only if  $\mathfrak{Y}^\# \mu \ll \mu$  and there exists  $\mathfrak{Z} : \mathfrak{N} \rightarrow \mathfrak{N}$  such that  $\mathfrak{Z} \circ \mathfrak{Y} = \text{Id}_{\mathfrak{N}}$ ,  $\mu$ -a.s. The map  $\mathfrak{Y} : \mathfrak{N} \rightarrow \mathfrak{N}$  is right invertible if and only if there exists  $\mathfrak{Z} : \mathfrak{N} \rightarrow \mathfrak{N}$  such that  $\mathfrak{Z}^\# \mu \ll \mu$  and  $\mathfrak{Y} \circ \mathfrak{Z} = \text{Id}_{\mathfrak{N}}$ ,  $\mu$ -a.s.  
The map  $\mathfrak{Y}$  is invertible if it is both left and right invertible.

**Lemma 4.1** *If there exists  $\mathfrak{Z}$  such that  $\mathfrak{Z} \circ \mathfrak{Y} = \text{Id}_{\mathfrak{N}}$ ,  $\mu$ -a.s. then  $\mathfrak{Y} \circ \mathfrak{Z} = \text{Id}_{\mathfrak{N}}$ ,  $\mathfrak{Y}^\# \mu$ -a.s. If additionally,  $\mathfrak{Y}^\# \mu$  is equivalent to  $\mu$  and  $\mathfrak{Z}^\# \mu \ll \mu$ , then  $\mathfrak{Y}$  is invertible and  $\mathfrak{Z}^\# \mu$  is equivalent to  $\mu$ .*

*Proof.* We have

$$\begin{aligned} \mathfrak{Y}^\# \mu(\mathfrak{Y} \circ \mathfrak{Z} = \text{Id}_{\mathfrak{N}}) &= \mu(\mathfrak{Y} \circ \mathfrak{Z} \circ \mathfrak{Y} = \mathfrak{Y}) \\ &= \mu(\mathfrak{Y} = \mathfrak{Y}) \\ &= 1. \end{aligned}$$

The first assertion follows. If the two measures  $\mathfrak{Y}^\# \mu$  and  $\mu$  are equivalent, then  $\mathfrak{Y} \circ \mathfrak{Z} = \text{Id}_{\mathfrak{N}}$   $\mu$ -almost-surely thus  $\mathfrak{Y}$  is invertible.

Let  $A$  such that  $\mathfrak{Z}^\# \mu(A) = 0$ . This means

$$\mathbf{E}_\mu[\mathbf{1}_A \circ \mathfrak{Z}] = 0.$$

Since  $\mathfrak{Y}^\# \mu$  is equivalent to  $\mu$ , we get

$$0 = \mathbf{E}_\mu[\mathbf{1}_A \circ \mathfrak{Z} \circ \mathfrak{Y}] = \mathbf{E}_\mu[\mathbf{1}_A]$$

hence  $\mu \ll \mathfrak{Z}^\# \mu$  and the equivalence follows.  $\square$

We begin by a technical lemma which indicates how the time changes of two inverse maps are related to each other.

**Lemma 4.2** *For  $y \in \mathcal{P}^{++}(\mathcal{F})$  and  $z$  a  $\pi$ -a.s. positive process such that  $z^*(Y, t)$  is an  $\mathcal{F}^{y^*}$ -stopping time, we have*

$$(\mathbf{Z} \circ \mathbf{Y})(t) = N\left(y^*(N, z^*(Y, t))\right). \quad (26)$$

*Then  $\mathbf{Z}$  is the left inverse of  $\mathbf{Y}$  if and only if*

$$z^*(Y, t) = y(N, t), \quad \forall t, \pi\text{-a.s.} \quad (27)$$

*or equivalently*

$$z(Y, t) = y^*(N, t), \quad \forall t, \pi\text{-a.s.} \quad (28)$$

*Proof.* The first part come from the identities:

$$\begin{aligned} (\mathbf{Z} \circ \mathbf{Y})(N)(t) &= Y\left(z^*(Y, t)\right) \\ &= N\left(y^*(N, z^*(Y, t))\right). \end{aligned}$$

Thus,

$$\mathbf{Z} \circ \mathbf{Y} = \text{Id} \iff y^*(N, z^*(Y, t)) = t, \forall t, \pi\text{-a.s.}$$

According to (12), this entails (27).  $\square$

We have already seen in Lemma 2.5 that  $\mathcal{N}^{y^*}$  is larger than  $\mathcal{Y}$ . Actually, there is a stronger result:

**Theorem 4.3** *Let  $y \in \mathcal{P}^{++}(\mathcal{N}, \pi)$ . Then,  $\mathcal{N}^{y^*} = \mathcal{Y}$  if and only if  $\mathbf{Y}$  admits a left inverse.*

*Proof.* If  $\mathbf{Y}$  is left-invertible then there exists  $z$  such that (28) holds :

$$z(Y, t) = y^*(N, t), \forall t, \pi\text{-a.s.}$$

This means that  $y^*(N, s)$  is  $\mathcal{Y}_s \subset \mathcal{Y}_t$  measurable, thus  $\mathcal{Y}_t \vee \sigma(y^*(s), s \leq t) = \mathcal{Y}_t$ .

Conversely, if  $\mathcal{N}_t^{y^*} = \mathcal{Y}_t$  then as  $y^*(N, t)$  is  $\mathcal{N}_t^{y^*}$  measurable, it is also  $\mathcal{Y}_t$ -measurable. Hence, for any  $t \geq 0$ , there exists a random variable  $\tilde{z}(Y^t, t)$  such that

$$\tilde{z}(Y^t, t) = y^*(N, t), \pi\text{-a.s.}$$

We can then find a full probability set  $A$  such that

$$\tilde{z}(Y^t(\omega), t) = y^*(\omega, t), \forall t \in \mathbf{Q}, \forall \omega \in A. \quad (29)$$

Let

$$z(Y^t(\omega), t) = \begin{cases} y^*(\omega, t) & \text{if } t \in \mathbf{Q} \\ \lim_{r_n \rightarrow t, r_n \in \mathbf{Q}} y^*(\omega, r_n) & \text{if } t \notin \mathbf{Q}. \end{cases}$$

By the sample-path continuity of  $y^*$ , (29) holds for any  $t \in \mathbf{R}^+$  with probability 1. Hence, Lemma 4.2 implies that  $\mathbf{Y}$  admits a left inverse.  $\square$

In the Introduction, we loosely defined g-Hawkes processes as point processes whose compensator depends on themselves. We stick to this definition up to now, until we define this notion properly in Section 6.

**Theorem 4.4** *Let  $y \in \mathcal{P}^{++}(\mathcal{N})$ . Then,  $\mathbf{Y}$  admits a right inverse of the form  $Z = N(z^*)$  where for any  $t > 0$ ,  $z^*(N, t)$  is an  $\mathcal{N}$ -stopping time if and only if there exists a g-Hawkes process  $Z$  with compensator  $y(Z, \cdot)$ , i.e. a solution of the equation*

$$Z(t) = N\left(y(Z, t)\right) \quad (30)$$

*with the additional constraint that  $y(Z, t)$  is an  $\mathcal{N}$ -stopping time.*

*Proof.* If  $Z = N(z^*)$  is the right inverse of  $\mathbf{Y}$ , then, according to (28), we have

$$z^*(N, t) = y(Z(N), t), \pi - \text{a.s.}$$

so that, if we set

$$Z(t) := N\left(z^*(N, t)\right),$$

we get

$$Z(t) = N\left(y(Z(N), t)\right).$$

This means that  $Z$  satisfies (30). Since  $z^*(N, t)$  is an  $\mathcal{N}$ -stopping time, so does  $y(Z, t)$  and the additional constraint is altogether satisfied.

Conversely, if  $Z$  satisfies (30) and  $y(Z, t)$  is an  $\mathcal{N}$ -stopping time for any  $t \geq 0$ , we set

$$z^*(N, t) = y(Z, t)$$

so that  $z^*(N, t)$  is a stopping time for any  $t \geq 0$ . Furthermore, according to (27) we have  $\mathbf{Y} \circ \mathbf{Z} = \text{Id}_{\mathfrak{Y}}$ , which means that  $\mathbf{Y}$  is right invertible.  $\square$

**Corollary 4.5** *If  $Z$  satisfy (30) and  $y(Z, t)$  is an  $\mathcal{N}$  stopping time for any  $t \geq 0$ , then*

$$\mathcal{Z}_t = \mathcal{N}_{y(Z, t)}$$

*where  $\mathcal{Z}$  is the  $\sigma$ -field generated by the sample-paths of  $Z$ . Furthermore,  $\mathbf{Z}^\# \pi = \pi_y$ .*

*Proof.* Actually, Theorem 4.4 implies that  $\mathbf{Z}$  is left invertible and the conclusion follows by Theorem 4.3. Consequently, according to (30),  $Z$  has  $y(Z, \cdot)$  as a  $(\pi, \mathcal{Z})$ -compensator. This means that the law of  $Z$  has the same compensator with respect to its minimal filtration as  $N$  under  $\pi_y$ . Theorem 2.1 then entails that  $\mathbf{Z}^\# \pi = \pi_y$ .  $\square$

**Corollary 4.6** *Let  $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$ . Assume that  $\mathbf{Y}$  is right invertible and let  $Z$  be the g-Hawkes process defined in Theorem 4.4. Then,  $Z$  admits the martingale representation property: for any  $F \in L^2(\mathfrak{Y} \rightarrow \mathbf{R}, \mathbf{Z}^\# \pi)$ , there exists  $v$  a  $\mathcal{Z}$ -predictable process such that,  $\pi$ -a.s., we have*

$$F \circ \mathbf{Z} = \mathbf{E}_{\mathbf{Z}^\# \pi} [F] + \int_0^\infty v(Z, s) (dZ(s) - dZ^P(s))$$

*where*

$$v(N, s) = u\left(Y, y(N, s)\right).$$

*Note that  $v$  satisfies*

$$\mathbf{E}_\pi \left[ \int_0^\infty v(Z, s)^2 dZ^P(s) \right] < \infty.$$

*Proof.* Since  $\mathbf{Z}$  is the right inverse of  $\mathbf{Y}$ ,  $\mathbf{Y}$  is the left inverse of  $\mathbf{Z}$ . According to Theorem 4.3,  $\mathcal{Z} = \mathcal{N}^{\mathbf{Z}^*}$ . By the very definition of right invertibility,  $\mathbf{Z}^\# \ll \pi$  hence for  $F$  a random variable on  $(\mathfrak{N}, \pi)$ , the random variable  $F_Z := F \circ \mathbf{Z}$  is well defined. When  $F_Z$  is  $\pi$ -square integrable, we know from the martingale representation property for the Poisson process that there exists a square integrable, predictable process  $u$  such that

$$F_Z = \mathbf{E}_\pi [F_Z] + \int_0^\infty u(s) (dN(s) - ds).$$

The relation (28) means that

$$T_k(N) = y(Z, T_k(Z)).$$

We obtain

$$\begin{aligned} \int_0^\infty u(s) dN(s) &= \sum_{k \geq 1} u(N, T_k(N)) \\ &= \sum_{k \geq 1} u(Y(Z), y(Z, T_k(Z))) \\ &= \int_0^\infty (u(Y, \cdot) \circ \mathbf{Z})(s) dZ(s). \end{aligned}$$

We know that  $Z$  has  $(\pi, \mathcal{N}^{\mathbf{Z}^*})$  compensator  $Z^p(t) = y(Z, t)$  hence this process is also the compensator of  $Z$  for the filtration  $\mathcal{Z}$ . The change of variable  $s = y(Z, r)$  yields

$$\begin{aligned} \int_0^\infty u(N, s) ds &= \int_0^\infty u(N, y(Z, r)) \dot{y}(Z, r) dr \\ &= \int_0^\infty v(Z, s) dZ^p(r). \end{aligned}$$

The proof is thus complete. □

By iteration, it is classical (see [9]) that we can deduce a chaos decomposition for square integrable functionals of  $Z$ : for any  $F$  as in there exists  $(v_k, k \geq 1)$  symmetric and square integrable kernels such that for any  $F$  as in Corollary 4.6,

$$\begin{aligned} F(Z) &= \mathbf{E}_\pi [F(Z)] \\ &+ \sum_{n=1}^\infty \int_{0 < s_1 < s_2 < \dots < s_n} v_n(Z, s_1, \dots, s_n) (dZ(s_1) - dZ^p(s_1)) \dots (dZ(s_n) - dZ^p(s_n)). \end{aligned}$$

■ **Remark 6** A similar result is established in [17]. The two formula share the same difficulty of the explicit computation of the kernels, which here amounts to construct the process  $Y$ . We are more or less in the same situation as for the fractional Brownian motion. In both situations, there is a reference process (the Brownian motion for the fBm, the Poisson process here) with which the process under study is constructed [9]. We can then use the Malliavin calculus tools for the driving process but it seems rather hard to have intrinsic chaos expansion depending only on the fBm or on the g-Hawkes process itself. ■

**Definition 4.2** For  $\mu$  and  $\nu$  two probability measures on  $\mathfrak{N}$ , the relative entropy of  $\nu$  with respect to  $\mu$  is given by

$$H(\nu|\mu) = \begin{cases} \mathbf{E}_\nu \left[ \log \left( \frac{d\nu}{d\mu} \Big|_{\mathcal{N}_\infty} \right) \right] & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

Following the proof of [20, Proposition 2.1], we have:

**Theorem 4.7** Let  $z \in \mathcal{P}^{++}(\mathcal{N})$ . Assume  $\pi_z$  is equivalent to  $\pi$  on  $\mathcal{N}_\infty$  and that there exists  $\dot{y} \in \mathcal{P}_2^{++}(\mathcal{N}, \pi)$  such that  $\mathbf{Z} \circ \mathbf{Y} = I_{\mathfrak{N}}$   $\pi$ -a.s. Then the following assertions are equivalent:

(i) We have

$$\mathbf{E}_\pi [\Lambda_y] = 1 \text{ and } \pi(\Lambda_y > 0) = 1. \quad (31)$$

(ii)  $\mathbf{Y}^\# \pi$  is equivalent to  $\pi$  on  $\mathcal{N}_\infty$ .

(iii)  $\mathbf{Z}$  is left invertible with inverse  $\mathbf{Y}$ .

(iv)  $\mathbf{Y}^\# \pi = \pi_z$  on  $\mathcal{N}_\infty$  and we have the following identity

$$\log(\Lambda_z \circ \mathbf{Y}) = \int_0^\infty \log(\dot{y}^*(N, s)) dY(s) + \int_0^\infty (1 - \dot{y}^*(N, s)) ds, \quad \pi - a.s. \quad (32)$$

*Proof.* (i)  $\implies$  (ii). According to Lemma 2.3,  $\pi_y$  is equivalent to  $\pi$  on  $\mathcal{N}_\infty$ . Apply Theorem 3.4 and point (ii) follows.

(ii)  $\implies$  (iii). According to Lemma 4.1,  $\mathbf{Y} \circ \mathbf{Z} = \text{Id}$ ,  $\mathbf{Y}^\# \pi$  a.s. In view of (ii), this identity holds  $\pi$  almost surely. Moreover, still according to (ii),

$$\mathbf{Z}^\# \pi \sim \mathbf{Z}^\# (\mathbf{Y}^\# \pi) = \pi$$

and (iii) follows from Lemma 4.1.

(iii)  $\implies$  (iv). It follows from (iii) that  $\pi_z = (\mathbf{Y} \circ \mathbf{Z})^\# \pi_z = \mathbf{Y}^\# (\mathbf{Z}^\# \pi_z)$ . Recall that the quasi-invariance theorem says that  $\mathbf{Z}^\# \pi_z = \pi$  hence  $\mathbf{Y}^\# \pi = \pi_z$ .

According to (17) and (18),

$$\Lambda_{z^*} \circ \mathbf{Y} = \exp \left( - \int_0^\infty \log(\dot{z}^*(Y, y^*(N, s))) dY(s) + \int_0^\infty (\dot{z}^*(Y, y^*(N, s)) - 1) \dot{y}^*(N, s) ds \right).$$

Since  $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{N}}$ , according to (27) we have

$$z^*(Y, y^*(N, t)) = t.$$

By differentiation, we obtain

$$\dot{z}^*(Y, y^*(N, t)) \times \dot{y}^*(N, t) = 1, \quad \pi - a.s. \quad (33)$$

We obtain

$$\log(\Lambda_z \circ \mathbf{Y}) = \int_0^\infty \log(\dot{y}^*(N, s)) dY(s) + \int_0^\infty \left( \frac{1}{\dot{y}^*(N, s)} - 1 \right) \dot{y}^*(N, s) ds$$

and (32) follows  
(iv) $\implies$ (i). Recall that

$$\Lambda_y = \exp\left(-\int_0^\infty \log(\dot{y}^*(N, s)) dY(s) - \int_0^\infty (1 - \dot{y}^*(N, s)) ds\right).$$

Equation (32) amounts to

$$\Lambda_y = \frac{1}{\Lambda_{z^*}} \circ \mathbf{Y} = \frac{1}{\Lambda_z} \circ \mathbf{Y}.$$

Since  $\mathbf{Y}^\# \pi = \pi_z$ , we obtain

$$\begin{aligned} \mathbf{E}_\pi [\Lambda_y] &= \mathbf{E}_{\pi_z} \left[ \frac{1}{\Lambda_z} \right] \\ &= \mathbf{E}_\pi \left[ \frac{1}{\Lambda_z} \Lambda_z \right] = 1. \end{aligned}$$

Moreover, still from (32), we deduce that

$$\begin{aligned} \pi(\Lambda_y = 0) &= \pi\left(\frac{1}{\Lambda_z} \circ \mathbf{Y} = 0\right) \\ &= \mathbf{Y}^\# \pi(\Lambda_z = +\infty) \\ &= \pi_z(\Lambda_z = +\infty). \end{aligned}$$

According to the hypothesis,  $\pi_z$  is equivalent to  $\pi$  and Lemma 2.3 entails that  $\mathbf{E}_\pi[\Lambda_z]$  is finite hence  $\pi(\Lambda_z = +\infty) = 0$  and then  $\pi_z(\Lambda_z = +\infty) = 0$ . It follows that

$$\pi(\Lambda_y = 0) = 0,$$

so that (i) holds. □

**Lemma 4.8** *Let  $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$  such that  $\mathbf{E}_\pi[\Lambda_y] = 1$  and*

$$\mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(N, s) - 1) ds \right] < \infty.$$

*Then,*

$$\mathbf{E}_\pi \left[ -\log \Lambda_{y^*} \right] \leq \mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}(N, s) - 1) ds. \right] \quad (34)$$

*Proof.* We have already seen that

$$\dot{y}^*(t) = \frac{1}{\dot{y}(y^*(t))} = \tau_{y^*} \left( \frac{1}{\dot{y}} \right) (t).$$

By hypothesis,  $\dot{y}$  is  $\mathcal{N}$ -predictable, thus, according to [19, Theorem 10.17(c)],  $\dot{y}^*$  is predictable with respect to the filtration  $\mathcal{N}^{y^*}$ .

From (23), we have

$$\begin{aligned} \mathbf{E} \left[ -\log \Lambda_{y^*}^*(t) \right] &= \mathbf{E} \left[ \int_0^t \log(y^*(s)) dY(s) + \int_0^t (1 - y^*(s)) ds \right] \\ &= \mathbf{E} \left[ \int_0^t y^*(s) \log(y^*(s)) + 1 - y^*(s) ds \right] \\ &\leq \mathbf{E} \left[ \int_0^\infty m(y^*(s) - 1) ds \right]. \end{aligned}$$

since  $Y$  has  $\mathcal{N}^{y^*}$  compensator  $y^*$  and  $\log y^*$  is  $\mathcal{N}^{y^*}$  predictable. It remains to prove that we can pass to the limit in the left-hand-side. Consider the non-negative, convex function  $\psi(x) = x - \log x$ . From Fatou's Lemma, we have

$$\begin{aligned} \mathbf{E} \left[ \psi(\Lambda_{y^*}^*) \right] &\leq \liminf_{t \rightarrow \infty} \mathbf{E} \left[ \psi(\Lambda_{y^*}^*(t)) \right] \\ &\leq 1 + \mathbf{E} \left[ \int_0^\infty m(y^*(s) - 1) ds \right]. \end{aligned}$$

This means that the non-negative submartingale  $(\psi(\Lambda_{y^*}^*(t)), t \geq 0)$  is uniformly integrable. From (22), we know that  $\Lambda_y = \Lambda_{y^*}^*$  hence in view of the first hypothesis, the non-negative martingale  $(\Lambda_{y^*}^*(t), t \geq 0)$  is uniformly integrable,

$$-\log \Lambda_{y^*}^*(t) \xrightarrow[t \rightarrow \infty]{L^1} -\log \Lambda_{y^*}^*.$$

This means that

$$1 + \mathbf{E} \left[ -\log \Lambda_{y^*}^* \right] \leq 1 + \mathbf{E} \left[ \int_0^\infty m(y^*(s) - 1) ds \right].$$

The proof is thus complete.  $\square$

**Theorem 4.9** *Let  $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$  such that  $\mathbf{E}_\pi[\Lambda_y] = 1$  and*

$$\mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right] < \infty.$$

*If  $\mathbf{Y}^\# \pi \ll \pi$ , we always have*

$$H(\mathbf{Y}^\# \pi | \pi) \leq \mathbf{E} \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right]. \quad (35)$$

*Moreover, the map  $\mathbf{Y}$  is left invertible if and only if*

$$H(\mathbf{Y}^\# \pi | \pi) = \mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right]. \quad (36)$$

■ **Remark 7** Following [2], the function  $x \mapsto m(x - 1)$  plays in the Poisson settings the exact same rôle as the function  $x \mapsto x^2/2$  in the Gaussian world. Since the entropic criterion in the Wiener reads as

$$H(U^\# \mu | \mu) = \frac{1}{2} \mathbf{E}_\mu \left[ \int_0^1 \dot{u}(s)^2 ds \right]$$

where  $U$  is defined in (2), we see that the Gaussian-Poisson analogy alluded to in [2] goes even further.  $\blacksquare$

*Proof.* Assume that  $\mathbf{Y}^\# \pi$  is absolutely continuous with respect to  $\pi$  on  $\mathcal{N}_\infty$ . According to Theorem 2.2, there exists  $\dot{z} \in \mathcal{D}_2^+(\mathcal{N}, \mathbf{Y}^\# \pi)$  such that

$$\left. \frac{d\mathbf{Y}^\# \pi}{d\pi} \right|_{\mathcal{N}_\infty} = \Lambda_z.$$

The quasi-invariance Theorem then says that for each  $t > 0$ , for  $f : \mathfrak{N} \rightarrow \mathbf{R}^+$  bounded and continuous,

$$\begin{aligned} \mathbf{E}_\pi [f \circ \mathbf{Y}^t] &= \mathbf{E}_\pi [f(Y^t)] \\ &= \mathbf{E}_\pi [f \Lambda_z(t)] \\ &= \mathbf{E}_\pi [f \circ \mathbf{Y}^t \Lambda_z \circ \mathbf{Y}^t(t) \Lambda_{y^*}^*(t)]. \end{aligned}$$

By Fatou's Lemma, we obtain

$$\mathbf{E}_\pi [f \circ \mathbf{Y} \Lambda_z \circ \mathbf{Y} \Lambda_{y^*}^*] \leq \mathbf{E}_\pi [f \circ \mathbf{Y}]$$

or equivalently

$$\mathbf{E}_\pi [f \circ \mathbf{Y} \Lambda_z \circ \mathbf{Y} \mathbf{E} [\Lambda_{y^*}^* | \mathcal{Y}_\infty]] \leq \mathbf{E}_\pi [f \circ \mathbf{Y}]$$

Hence,  $\pi$ -a.s. we have

$$\Lambda_z \circ \mathbf{Y} \times \mathbf{E}_\pi [\Lambda_{y^*}^* | \mathcal{Y}_\infty] \leq 1. \quad (37)$$

It follows that

$$\begin{aligned} 0 \leq H(\mathbf{Y}^\# \pi | \pi) &= \mathbf{E}_\pi [\log \Lambda_z \circ \mathbf{Y}] \\ &\leq -\mathbf{E}_\pi [\log \mathbf{E}_\pi [\Lambda_{y^*}^* | \mathcal{Y}_\infty]]. \end{aligned} \quad (38)$$

Since  $-\log$  is convex, the Jensen inequality stands that

$$\begin{aligned} H(\mathbf{Y}^\# \pi | \pi) &\leq -\mathbf{E}_\pi [\log \Lambda_{y^*}^*] \\ &= \mathbf{E}_\pi \left[ \int_0^\infty m(y^*(s) - 1) ds \right], \end{aligned} \quad (39)$$

according to Lemma 4.8. Then the first part holds.

Assume now that (36) holds. Then (38) and (39) are indeed equalities. On the one hand, this means that we have equality in the Jensen inequality used to derive (39). Since  $-\log$  is strictly convex, it follows that [23, Cor. 2.1]:

$$\mathbf{E}_\pi [\Lambda_{y^*}^* | \mathcal{Y}_\infty] = \Lambda_{y^*}^*, \quad \pi - \text{a.s.}$$



On the other hand, this also implies that (38) is an equality and as a consequence of (37), we obtain

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = -\log \Lambda_{y^*}^*. \quad (40)$$

According to (17) and (18),

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = -\int_0^\infty \log(\dot{z}^*(Y, y^*(N, s))) dY(s) + \int_0^\infty (\dot{z}^*(Y, y^*(N, s)) - 1) \dot{y}^*(N, s) ds$$

and

$$-\log \Lambda_{y^*}^* = \int_0^\infty \log(\dot{y}^*(N, s)) dY(s) - \int_0^\infty (\dot{y}^*(N, s) - 1) ds.$$

Hence, according to (36), we can take the conditional expectation with respect to  $\mathcal{Y}_t$  in both equalities to obtain:

$$\begin{aligned} -\int_0^t \log(\dot{z}^*(Y, y^*(N, s))) dY(s) + \int_0^t (\dot{z}^*(Y, y^*(N, s)) - 1) \dot{y}^*(N, s) ds \\ = \int_0^t \log(\dot{y}^*(N, s)) dY(s) - \int_0^t (\dot{y}^*(N, s) - 1) ds. \end{aligned}$$

Equating the jumps yields

$$\dot{z}^*(Y, y^*(N, s)) \times \dot{y}^*(N, s) = 1,$$

then, by integration, we get

$$z^*(Y, y^*(N, t)) = t.$$

Furthermore,

$$(z^*(Y, t) \leq s) = (y(N, t) \leq s) = (y^*(N, s) \geq t) \in \mathcal{N}_s^{y^*},$$

hence  $z^*(Y, t)$  is an  $\mathcal{N}^{y^*}$  stopping time. According to (27), this means that  $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\eta_t}$  and then  $\mathbf{Y}$  is left invertible.

Conversely, if  $\mathbf{Y}$  is left invertible. According to the Definition 4.1,  $\mathbf{Y}^\# \pi$  is absolutely continuous with respect to  $\pi$ . Let us denote by  $\mathfrak{Z}$  the map such that

$$\mathfrak{Z} \circ \mathbf{Y} = \text{Id}_{\eta_t}, \pi - \text{a.s.}$$

Define the process  $z$  by

$$z(N, t) = y^*(\mathfrak{Z}(N), t), \forall t \geq 0.$$

Since  $\mathbf{Y}^\# \pi \ll \pi$ ,

$$\mathbf{Y}^\# \pi(z(N, \cdot) = y^*(\mathfrak{Z}(N), \cdot)) = 1$$

and

$$\pi(z(Y, \cdot) = y^*(N, \cdot)) = \mathbf{Y}^\# \pi(z(N, \cdot) = y^*(\mathfrak{Z}(N), \cdot)).$$

This means that

$$z(Y, t) = y^*(N, t), \forall t \geq 0, \pi\text{-a.s.}$$

or otherwise stated,  $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{N}}$ ,  $\pi$ -a.s. Moreover, by differentiation, we get

$$\dot{z}^*(Y, y^*(N, t)) = \frac{1}{\dot{y}^*(N, t)}, \quad \forall t \geq 0, \pi - \text{a.s.} \quad (41)$$

Since  $\dot{y}(N, \cdot)$  belongs to  $\mathcal{P}_2^{++}(\mathcal{N}, \pi)$ , so does  $\dot{z}^*(Y, \cdot)$ , hence we can write

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = - \int_0^\infty \log(\dot{z}^*(Y, y^*(N, s))) dY(s) + \int_0^\infty (\dot{z}^*(Y, y^*(N, s)) - 1) \dot{y}^*(N, s) ds,$$

and (41) entails that

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = - \log \Lambda_{y^*}^*. \quad (42)$$

Let

$$R = \frac{d\mathbf{Y}^\# \pi}{d\pi}.$$

For any  $f : \mathfrak{N} \rightarrow \mathbf{R}$  continuous and bounded, for any  $t > 0$ , we have

$$\begin{aligned} \mathbf{E}_\pi [fR] &= \mathbf{E}_\pi [f \circ \mathbf{Y}] \\ &= \mathbf{E}_\pi [(f \Lambda_{z^*}^*) \circ \mathbf{Y} \Lambda_{y^*}^*] \\ &= \mathbf{E}_\pi [f \Lambda_{z^*}^*] \end{aligned}$$

according to (42) and to the quasi-invariance Theorem. It follows that  $R = \Lambda_{z^*}^*$ ,  $\pi$ -a.s. Plug this identity into (42) to obtain

$$\begin{aligned} H(\mathbf{Y}^* \pi | \pi) &= \mathbf{E}_\pi [\log R \circ \mathbf{Y}] \\ &= \mathbf{E}_\pi [\log \Lambda_{z^*}^* \circ \mathbf{Y}] \\ &= \mathbf{E}_\pi [-\log \Lambda_{y^*}^*] \\ &= \mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right]. \end{aligned}$$

The entropic criterion is thus satisfied.  $\square$

If we restrict even further the set of time changes, we can construct a g-Hawkes process driven by these time changes and thus obtain the invertibility (see [15, 25] for the Brownian analog).

**Corollary 4.10** *Let  $\dot{y} \in \mathcal{P}_\infty^{++}(\mathcal{N}, \pi)$  be piecewise constant: If we denote by  $T$  the time after which  $\dot{y}(s) = 1$ , consider a partition of  $[0, T]$ ,  $0 = t_0 < t_1 < \dots < t_k = T < t_{k+1} = +\infty$  and assume that there exist  $\alpha_0 \in \mathbf{R}^+$  and some random variables  $(\alpha_j, j = 1, \dots, k)$  such that for some  $\epsilon > 0$ ,*

$$\epsilon \leq \alpha_j(N) \leq 1/\epsilon, \quad \forall j = 0, \dots, k$$

and

$$\dot{y}(N, s) = \alpha_0 \mathbf{1}_{(0, t_1]}(s) + \sum_{j=1}^{k-1} \alpha_j(N^{t_j}) \mathbf{1}_{(t_j, t_{j+1}]}(s) + \alpha_k(N^T) \mathbf{1}_{[T, \infty)}(s).$$

*Then,  $\mathbf{Y}$  is invertible.*

*Proof.* We first prove that  $\mathbf{Y}$  is right invertible.

The g-Hawkes process  $Z$  is constructed inductively. On  $[0, t_1]$ , we set

$$Z(t) = N(\alpha_0 t).$$

Then,

$$y(Z, t) = \alpha_0 t$$

and we do have  $Z(t) = N(y(Z, t))$ . For any  $t \leq t_1$ ,  $y(N, t)$  is deterministic hence it is an  $\mathcal{N}$ -stopping time.

Assume that  $Z$  is constructed on  $[0, t_m]$  with  $m < k$  and  $y(Z, t)$  is an  $\mathcal{N}$ -stopping time for  $t \leq t_m$ . For  $t \in [t_m, t_{m+1}]$ , we have

$$y(Z, t) = y(Z, t_m) + \alpha_m(Z^{t_m})(t - t_m). \quad (43)$$

By the induction hypothesis,  $y(Z, t_m)$  is an  $\mathcal{N}$ -stopping time hence the  $\sigma$ -field  $\mathcal{N}_{y(Z, t_m)}$  is well defined and  $y(Z, t_m)$  belongs to this  $\sigma$ -field. Furthermore, for  $t \leq t_m$ ,

$$Z(t) = N(y(Z, t)) \in \mathcal{N}_{y(Z, t_m)}.$$

It follows that for  $t \in [t_m, t_{m+1}]$ ,

$$y(Z, t) \in \mathcal{N}_{y(Z, t_m)}.$$

Since  $\alpha_m > 0$ , for any  $s \geq 0$ , according to (43),

$$\begin{aligned} \left( y(Z, t) \leq s \right) &= \left( \alpha_m(Z^{t_m}) \leq \frac{s - y(t_m)}{t - t_m} \right) \\ &= \left( \alpha_m(Z^{t_m}) \leq \frac{s - y(t_m)}{t - t_m} \right) \cap (y(Z, t_m) \leq s) \end{aligned}$$

which belongs to  $\mathcal{N}_s$  by the definition of  $\mathcal{N}_{y(Z, t_m)}$ . Hence  $y(Z, t)$  is an  $\mathcal{N}$ -stopping time for  $t \leq t_{m+1}$ . Moreover, (43) guarantees that there is no ambiguity to define  $Z$  on  $[t_m, t_{m+1}]$  by  $Z(t) = N(y(Z, t))$ . According to Theorem 4.4,  $\mathbf{Y}$  is right invertible.

Remark that  $Z$  has  $(\mathcal{N}, \pi)$  compensator  $y(Z, \cdot)$ . In view of Theorem 2.2,

$$\left. \frac{d\pi_z}{d\pi} \right|_{\mathcal{N}_t} = \Lambda_{y(Z, t)}.$$

From the form of  $y$ , it is clear that  $\Lambda_{y(Z, t)} > 0$  for all  $t \geq 0$  and that

$$\left( \Lambda_{y(Z, t)}, t \geq 0 \right)$$

is uniformly integrable. Point (i) of Theorem 4.7 then entails that  $\mathbf{Y}$  is left invertible and then invertible.  $\square$

## 5 Variational representation of the entropy

Let

$$\begin{aligned}\mathcal{D}_m^{++} &= \{y, \dot{y} \in \mathcal{D}^{++}(\mathcal{N}) \text{ and } (\dot{y} - 1) \in L^1(\mathfrak{N}; L_m, \pi)\} \\ \mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi) &= \mathcal{D}_{\infty}^{++}(\mathcal{N}, \pi) \cap \{y, \dot{y} \text{ piecewise constant}\} \\ \mathfrak{M}_m(\mathfrak{N}) &= \{\mu, \exists y \in \mathcal{D}_m^{++} \text{ such that } \mu = Y^\# \pi\}.\end{aligned}$$

The theorem reads as follows:

**Theorem 5.1 — Variational representation of the entropy.** *Let  $f : \mathfrak{N} \rightarrow \mathbf{R}$  such that*

$$\mathbf{E}_\pi \left[ |f|(1 + e^f) \right] < \infty.$$

*Then,*

$$\log \mathbf{E}_\pi \left[ e^f \right] = \sup_{y \in \mathcal{D}_m^{++}} \left( \mathbf{E}_\pi [f(N \circ y^*)] - \mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right] \right).$$

*Proof.* The duality between the relative entropy and the logarithmic Laplace transform says that

$$\log \mathbf{E}_\pi \left[ e^f \right] = \sup_{\mu \in \mathfrak{M}^1(\mathfrak{N})} \left( \int_{\mathfrak{N}} f d\mu - H(\mu | \pi) \right) \quad (44)$$

where  $\mathfrak{M}^1(\mathfrak{N})$  is the set of probability measures on  $\mathfrak{N}$  which are absolutely continuous with respect to  $\pi$  on  $\mathcal{N}_\infty$ . Furthermore, the supremum is attained at the measure  $\mu_f$  whose  $\pi$ -density is given by

$$\frac{d\mu_f}{d\pi} = \frac{e^f}{\mathbf{E}_\pi [e^f]}.$$

In view of (44), we evidently have

$$\log \mathbf{E}_\pi \left[ e^f \right] \geq \sup_{\mu \in \mathfrak{M}_m(\mathfrak{N})} \left( \int_{\mathfrak{N}} f d\mu - H(\mu | \pi) \right).$$

According to Corollary 4.10, for  $y \in \mathcal{D}_m^{++}$

$$\int_{\mathfrak{N}} f d\mu - H(Y^\# \pi | \pi) \geq \mathbf{E}_\pi [f(N \circ y^*)] - \mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right].$$

Since  $\mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi) \subset \mathcal{D}_m^{++}$ , the entropic criterion implies that

$$\begin{aligned}\sup_{\mu \in \mathfrak{M}^1(\mathfrak{N})} \int_{\mathfrak{N}} f d\mu - H(\mu | \pi) &\geq \sup_{y \in \mathcal{D}_m^{++}} \mathbf{E}_\pi [f(N \circ y^*)] - \mathbf{E}_\pi \left[ \int_0^\infty m(\dot{y}^*(s) - 1) ds \right] \\ &\geq \sup_{y \in \mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)} \mathbf{E}_\pi [f(N \circ y^*)] - H(Y^\# \pi | \pi).\end{aligned}$$

It remains to prove that we can find  $(\dot{y}_n, n \geq 1)$ , a sequence of elements of  $\mathcal{P}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)$  such that

$$\begin{aligned} \int_{\mathfrak{N}} f \, d(\mathbf{Y}_n^\# \pi) &\xrightarrow{n \rightarrow \infty} \int_{\mathfrak{N}} f \, d\mu_f \\ H(\mathbf{Y}_n^\# \pi | \pi) &\xrightarrow{n \rightarrow \infty} H(\mu | \pi) \end{aligned}$$

to conclude. This is the object of the next theorem. □

**Theorem 5.2** *Let  $\nu$  be a probability measure on  $\mathfrak{N}$  absolutely continuous with respect to  $\pi$  and*

$$L = \frac{d\nu}{d\pi}.$$

*Assume that*

$$\begin{aligned} L \log L &\in L^1(\pi) \\ \log L &\in L^r(\pi) \end{aligned}$$

*for some  $r > 1$ . Then, there exists  $(\dot{y}_n, n \geq 1)$  a sequence of elements of  $\mathcal{P}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)$  such that*

$$L_n \log L_n \xrightarrow[n \rightarrow \infty]{L^1(\pi)} L \log L \tag{45}$$

*and*

$$L_n \log L \xrightarrow[n \rightarrow \infty]{L^1(\pi)} L \log L \tag{46}$$

*where*

$$L_n = \frac{d\mathbf{Y}_n^\# \pi}{d\pi}.$$

*Proof.* We first show that we can suppose  $L$  lower and upper bounded. Consider

$$\Phi_n = (L \wedge n) \vee \frac{1}{n}.$$

We have

$$|\Phi_n| \leq L + 1,$$

hence by dominated convergence,  $\Phi_n$  converges in  $L^1(\pi)$  to  $L$  and in particular,

$$\mathbf{E}_\pi[\Phi_n] \xrightarrow{n \rightarrow \infty} \mathbf{E}_\pi[L] = 1.$$

Let

$$L_n = \frac{\Phi_n}{\mathbf{E}_\pi[\Phi_n]}.$$

For any  $\alpha \in (0, 1)$ , for  $n$  sufficiently large,  $\mathbf{E}_\pi[\Phi_n] \geq \alpha$ . Moreover, for  $x \geq 0$ , we have

$$|x \log(x)| \leq \frac{1}{e} + \left| \frac{x}{\alpha} \log\left(\frac{x}{\alpha}\right) \right| \mathbf{1}_{x \geq \alpha}.$$

Hence,

$$|L_n \log L_n| \leq \frac{1}{e} + \left| \frac{L}{\alpha} \log \left( \frac{L}{\alpha} \right) \right|.$$

By dominated convergence again,  $L_n \log L_n$  converges to  $L \log L$  in  $L^1(\pi)$ . Similarly,

$$|L_n \log L| \leq \left| \frac{L}{\alpha} \log L \right|$$

and  $L_n \log L$  converges to  $L \log L$  in  $L^1(\pi)$ .

Assume now that  $L$  is lower and upper bounded by respectively  $m$  and  $M$ . We know that there exists  $\dot{y} \in \mathcal{D}^{++}(\mathcal{N}, \pi)$  such that

$$L = \Lambda_{\dot{y}}.$$

Set

$$L_n = \Lambda_{1+(\dot{y}^{n-1})} = \mathbf{E}_{\pi} [L | \mathcal{N}_n].$$

Since  $L$  is bounded, it is clear that (45) and (46) hold. We can then assume that there exists  $T > 0$  such that  $\dot{y}(s) = 1$  for  $s \geq T$ .

Moreover, from the Malliavin calculus for Poisson process, we know (see [10]) that

$$\dot{y}(s) = \frac{\mathbf{E}_{\pi} [D_s L | \mathcal{N}_s]}{\mathbf{E}_{\pi} [L | \mathcal{N}_s]}$$

where  $D_s F(N) = F(N + \epsilon_s) - F(N)$ . We then have

$$0 \leq \dot{y}(s) \leq \frac{M}{m}.$$

Consider  $\dot{y}_n = \dot{y} \vee n^{-1}$ , it is straightforward that (45) and (46) hold.

Finally, assume that  $\dot{y}$  is lower and upper bounded on some interval  $[0, T]$  and equal to 1 above  $T$ . Set

$$\begin{aligned} \dot{y}_n(s) &= 0 \text{ if } s \in [0, T/n) \\ \dot{y}_n(s) &= n \int_{(i-1)/n}^{i/n} \dot{y}(s) \, ds \text{ if } s \in [iT/n, (i+1)T/n) \end{aligned}$$

for  $i \in \{1, \dots, n-1\}$ . We see that  $\dot{y}_n$  belongs to  $\mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)$ . We know (see [24]) that  $\dot{y}_n$  converges in  $L^2(\mathfrak{N} \times [0, T], \pi \otimes ds)$  to  $\dot{y}$ . Moreover, it is easy to see that

$$\sup_n \mathbf{E}_{\pi} [\Lambda_{\dot{y}_n}^p] < \infty \text{ for any } p \geq 1.$$

Thus, (45) and (46) hold. □

## 6 Weak and strong g-Hawkes processes

**Definition 6.1** A filtered probability space is a triplet  $(\Omega, \mathcal{F}, \mathbf{P})$  where  $\Omega$  is a space, equipped with a right-continuous filtration  $\mathcal{F}$  and a probability  $\mathbf{P}$ .

In what follows, we equip  $\mathfrak{N}$  with the minimum filtration:

$$\mathcal{N}_t = \sigma \{ \omega([0, s]), 0 \leq s \leq t \}.$$

The process  $y : \mathfrak{N} \rightarrow \mathbf{R}$  is supposed to be  $\mathcal{N}$ -predictable and  $y$  is positive. We define the different notions of solution associated to the generalized Hawkes problem associated to  $y$ .

**Definition 6.2 — g-Hawkes problem.** Consider  $(\Omega, \mathcal{F}, \mathbf{P})$  a filtered probability space. By a solution of g- $H_y$ , we mean a couple of processes  $Y = (Z, N)$  such that

1. With probability one,  $Z$  and  $N$  are point processes in the sense of Definition 2.1,
2. For any  $t \geq 0$ , the random variable  $y(Z, t)$  is a  $\mathcal{F}$  stopping time,
3. The process  $(N(t) - t, t \geq 0)$  is a  $\mathcal{F}$ -local martingale,
4. The processes  $Z$  and  $N$  satisfy,  $\mathbf{P}$ -a.s. for any  $t \geq 0$ ,

$$Z(t) = N(y(Z, t)). \quad (47)$$

■ **Remark 8** Note that item 1. of Definition 6.2 and (47) imply that

$$y(\{T_1(Z), \dots, T_{q-1}(Z)\}, T_q(Z)) = T_q(N), \quad \forall q \geq 1. \quad (48)$$

However, (48) does not imply immediately (47) as it is not clear that  $y(Z, t)$  is an  $\mathcal{N}$  stopping time and thus that the application  $t \mapsto N(y(Z, t))$  is measurable. ■

**Definition 6.3 — Weak and strong solutions.** If we must specify the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  then the solution is said to be weak.

If we can find a solution for any filtered probability space on which we can construct a unit rate Poisson process, then the solution is said to be strong.

**Theorem 6.1** Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a filtered probability space and  $N$  an  $\mathcal{F}$ -adapted unit rate Poisson process. For  $y \in \mathcal{P}^{++}(\mathcal{F})$ , there exists at most one point process  $Z$  which satisfies (47).

*Proof.* If  $X$  and  $Y$  are two solutions of g- $H_y$  on the same probability space, according to (48), we have

$$X(t) = N(y(X, t)) \text{ and } Y(t) = N(y(Y, t)).$$

Since they both are point processes, we can speak of their jump times  $(T_q(X), q \geq 1)$  and  $(T_q(Y), q \geq 1)$ . We have

$$y(\emptyset, T_1(X)) = T_1(N) \text{ and } y(\emptyset, T_1(Y)) = T_1(N).$$

Since  $y(N, \cdot)$  is an homeomorphism,  $T_1(X) = T_1(Y)$ . Suppose proved that

$$T_j(X) = T_j(Y) \text{ for } j = 1, \dots, q-1.$$

Since  $y$  is predictable

$$y\left(\sum_{j=1}^{q-1} \epsilon_{T_j(X)}, T_q(X)\right) = T_q(N) = y\left(\sum_{j=1}^{q-1} \epsilon_{T_j(X)}, T_q(Y)\right).$$

Hence  $T_q(X) = T_q(Y)$  and then  $X = Y$ . Thus the uniqueness holds.  $\square$

■ **Remark 9** Actually, we have used in this proof the construction inherited from Algorithm 7.4.III of [7] to simulate a point process when its intensity is known. The trick is here is that due to the predictability of the compensator, we can calculate its new value after each jump time. That is why we need to know a priori that the process we are looking for is a point process and does not evolve between jump times.

The usual algorithm which is based on the thinning of a Poisson measure gives the sample-path of  $Z$  up to a given time and suffers from the rejection of a possibly large number of points. This algorithm yields the number of jumps we desire and may suffer from numerical instability in the root findings. ■

**Definition 6.4 — Martingale problem.** Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be given and  $Z$  a point process. If the following conditions are satisfied:

- i)  $y^*(Z, t)$  is  $\mathcal{F}$ -adapted,
- ii) The process

$$t \mapsto Z(y^*(Z, t)) - t$$

is a  $\mathcal{F}$ -local martingale,

we then say that  $(Z, \mathcal{F})$  satisfies the g-Hawkes martingale problem, denoted by  $g\text{-H}_y^m$ .

The first theorem is the exact analog of what happens for stochastic differential equations driven by a Brownian motion.

**Theorem 6.2 — Equivalence of weak and martingale solutions.** Let  $\dot{y} \in \mathcal{D}^{++}(\mathcal{F})$ . There exists a weak solution to  $g\text{-H}_y$  if and only if there exists a solution to  $g\text{-H}_y^m$ .

*Proof.* Assume that there exists  $(\Omega, \mathcal{F}, \mathbf{P})$  and  $Y = (Z, N)$  a solution to  $g\text{-H}_y$ . We assumed that  $\dot{y} > 0$ , thus

$$(y^*(Z, t) \leq s) = (y(Z, s) \geq t) \in \mathcal{F}_t,$$

since  $y(Z, t)$  is an  $\mathcal{F}$ -stopping time. Thus the process  $(y^*(Z, t), t \geq 0)$  is  $\mathcal{F}$ -adapted. Moreover, as  $y^*(Z, \cdot)$  is an homeomorphism on  $\mathbf{R}^+$ , we have,  $\mathbf{P}$ -a.s.,

$$Z(y^*(Z, t)) = N(t), \quad \forall t \geq 0,$$

and then

$$t \mapsto Z(y^*(Z, t))$$



is an  $\mathcal{F}$ -Poisson process of intensity 1, i.e. point ii) holds and  $(Z, \mathcal{F})$  solves  $\text{g-H}_y^m$ .

Conversely, if  $(\Omega, \mathcal{F}, \mathbf{P}, Z)$  solves  $\text{g-H}_y^m$ , this means that  $N$  defined by

$$N(t) := Z(y^*(Z, t))$$

is a unit rate Poisson process with respect to the filtration  $\mathcal{F}$  and taking the inverse of  $y^*(Z, \cdot)$ , we have

$$Z(t) = N(y(Z, t)),$$

thus (47) is satisfied. Moreover, from i) we deduce that

$$(y(Z, t) \leq s) = (y^*(Z, s) \geq t) \in \mathcal{F}_s.$$

Hence Point 2 of Definition 6.2 is satisfied and we can then say that  $(Z, N)$  satisfies  $\text{g-H}_y$  on  $(\Omega, \mathcal{F}, \mathbf{P})$  where  $N$  is defined by (6).  $\square$

**Definition 6.5 — Pathwise uniqueness.** We say that path-wise uniqueness holds for  $\text{g-H}_y$  whenever for any two solutions  $Y = (Z, N)$  and  $Y' = (Z', N')$  defined on the same filtered probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , we have  $Z = Z'$ , up to  $\mathbf{P}$ -indistinguishability.

**Definition 6.6 — Weak Uniqueness.** We say that weak uniqueness holds for  $\text{g-H}_y$  whenever for any two solutions  $Y = (Z, N)$  and  $Y' = (Z', N')$ , possibly defined on different probability spaces, the law of  $Z$  and  $Z'$  on the space  $(\mathfrak{N}, \mathcal{N}_\infty)$  do coincide.

For any  $k \geq 1$ , on  $\mathfrak{N}^k$ , we consider the  $\sigma$ -field  $\mathcal{N}^{\otimes(k)}$  defined by

$$\mathcal{N}^{\otimes(k)} = \otimes_{i=1}^k \mathcal{N}^i \text{ where } \mathcal{N}^i = \sigma(\eta_i(s), s \geq 0).$$

For any  $t \geq 0$ , we introduce the sub- $\sigma$  field

$$\mathcal{N}_t^{\otimes(k)} = \otimes_{i=1}^k \mathcal{N}_t^i.$$

For any solution  $Y$  of  $\text{g-H}_y$ , let  $\mu$  be the law of  $Y$  on  $(\mathfrak{N}^2, \mathcal{N}^{\otimes(2)})$ , i.e.  $\mu = Y^\# \mathbf{P}$ . Let  $\mu_{\eta_2}(d\eta_1)$  be the regular conditional distribution of  $\mu(d\eta_1, d\eta_2)$  given  $\eta_2$ :

1. For each  $\eta_2$ ,  $\mu_{\eta_2}(d\eta_1)$  is a probability measure on  $(\mathfrak{N}, \mathcal{N}^1)$ ,
2. For each  $B \in \mathcal{N}^1$ ,  $\mu_{\eta_2}(B)$  is  $\mathcal{N}^2$ -measurable in  $\eta_2$ ,
3. For any  $B \in \mathcal{N}^1, B' \in \mathcal{N}^2$ ,

$$\mu(B \times B') = \int_{B'} \mu_{\eta_2}(B) d\pi(\eta_2).$$

Let  $\mu_{\eta_2}^t$  be the regular conditional distribution of  $\mu$  given  $\mathcal{N}_t$ :

1. For each  $\eta_2 \in \mathfrak{N}$ ,  $\mu_{\eta_2}^t$  is a measure on  $\mathfrak{N}$ ,
2. for each  $B \in \mathcal{N}$ , the random variable  $\mu_{\eta_2}^t(B)$  is  $\mathcal{N}_t / \mathcal{B}(\mathbf{R}^+)$ -measurable,

3. for any  $B' \in \mathcal{N}_t$ ,

$$\mu(B \times B') = \int_{B'} \mu_{\eta_2}^t(B) \, d\pi(\eta_2).$$

**Lemma 6.3** For  $B \in \mathcal{N}_{y^*(\eta_1, t)}^1$ , the map  $\eta_2 \mapsto \mu_{\eta_2}(B)$  is  $\mathcal{N}_t^2 / \mathcal{B}(\mathbf{R}^+)$ -measurable.

*Proof.* We first prove that for  $B \in \mathcal{N}_{y^*(\eta_1, t)}^1$ , there exists  $\theta$  measurable, such that

$$\mathbf{1}_B(\eta_1) = \theta(\eta_2^t), \quad \mu\text{-a.s.} \quad (49)$$

or equivalently,  $\mathbf{1}_B(\eta_1) \in \mathcal{N}_t^2$ . According to [19], we know that  $\mathcal{N}_{y^*(\eta_1, t)}^1$  is generated by sets of the form

$$\bigcap_{j=0}^q \left( T_j(\eta_1) \leq a_j \right) \cap \left( T_q(\eta_1) < y^*(\eta_1, t) \leq T_{q+1}(\eta_1) \right) \quad (50)$$

for some  $a_j \in \mathbf{R}^+$  and some  $q \in \mathbf{N}$ . Recall that (47) implies that

$$\mu\left(\eta_1(t) = \eta_2(y(\eta_1, t)), \forall t \geq 0\right) = 1.$$

This means that

$$y(\eta_1, T_k(\eta_1)) = T_k(\eta_2) \iff T_k(\eta_1) = y^*(\eta_1, T_k(\eta_2)).$$

Hence for a set  $B$  of the form (50), we have

$$\begin{aligned} B &= \bigcap_{j=0}^q \left( y^*(\eta_1, T_j(\eta_2)) \leq a_j \right) \cap \left( T_q(\eta_2) < t \leq T_{q+1}(\eta_2) \right) \\ &= \bigcap_{j=0}^q \left( T_j(\eta_2) \leq y(\eta_1, a_j) \right) \cap \left( T_q(\eta_2) < t \leq T_{q+1}(\eta_2) \right) \end{aligned}$$

By definition of the g-Hawkes problem,  $y(\eta_1, a)$  is a  $\mathcal{N}^2$ -stopping time hence for any  $j \leq q$ ,

$$\left( T_j(\eta_2) \leq y(\eta_1, a_j) \right) \in \mathcal{N}_{T_j(\eta_2) \wedge y(\eta_1, a_j)}^2 \subset \mathcal{N}_{T_j(\eta_2)}^2 \subset \mathcal{N}_{T_q(\eta_2)}^2.$$

Hence  $B \in \mathcal{N}_t^2$ .

Let  $B \in \mathcal{N}_{y^*(\eta_1, t)}^1$ . For the second step of the proof, we have to prove that for  $F : \mathfrak{N} \rightarrow \mathbf{R}$  measurable and bounded

$$\int_{\mathfrak{N} \times \mathfrak{N}} F(\eta_2) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) = \int_{\mathfrak{N} \times \mathfrak{N}} F(\eta_2) \mu_{\eta_2}^t(B) \mu(d\eta_1, d\eta_2).$$

Since  $\eta_2$  is a Poisson process, for any  $F$  bounded, the martingale representation theorem valid for Poisson processes says that there exists  $u_F$  which is  $\mathcal{N}$ -adapted such that

$$\mathbf{E} \left[ \int_0^\infty u_F(\eta_2, s)^2 \, ds \right] < \infty$$

and

$$F(\eta_2) = \mathbf{E}[F] + \int_0^\infty u_F(\eta_2, s) \, d\tilde{\eta}_2(s)$$

where  $\tilde{\eta}_2(t) = \eta_2(t) - t$ . The process

$$(\eta_2, t) \longmapsto \int_0^t u_F(\eta_2, s) d\tilde{\eta}_2(s)$$

is a square integrable martingale with respect to the filtration  $\mathcal{N}$  and thus it is also a square martingale with respect to the filtration  $\mathcal{N}^{\otimes(2)}$ . Now then we have

$$\begin{aligned} & \int_{\mathfrak{N} \times \mathfrak{N}} F(\eta_2) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) = \mathbf{E}[F] \int_{\mathfrak{N} \times \mathfrak{N}} \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) \\ & + \int_{\mathfrak{N} \times \mathfrak{N}} \left( \int_t^\infty u_F(\eta_2, s) d\tilde{\eta}_2(s) \right) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) + \int_{\mathfrak{N} \times \mathfrak{N}} \left( \int_0^t u_F(\eta, s) d\tilde{\eta}(s) \right) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2). \end{aligned}$$

By the first part of the proof,  $\mathbf{1}_B(\eta_1) = \theta(\eta_2^t)$   $\mu$ -a.s. Hence, by the martingale property of the stochastic integral, the median term is null. We thus get

$$\begin{aligned} \int_{\mathfrak{N} \times \mathfrak{N}} F(\eta) \mathbf{1}_B(z) \mu(dz, d\eta) &= \int_{\mathfrak{N}} \left[ \mathbf{E}[F] + \left( \int_0^t u_F(\eta, s) d\tilde{\eta}(s) \right) \right] \mu_\eta^t(B) d\pi(\eta) \\ &= \int_{\mathfrak{N}} F(\eta) \mu_\eta^t(B) d\pi(\eta) \end{aligned}$$

by the same kind of reasoning. □

**Theorem 6.4 — Yamada-Watanabe.** *The following two properties hold.*

1. *Path-wise uniqueness implies weak uniqueness.*
2. *Moreover, if there exists a solution  $Y = (Z, N)$  of  $g$ - $H_y$ , on some  $(\Omega, \mathcal{F}, \mathbf{P})$  and that path-wise uniqueness holds then there exists  $F : \mathfrak{N} \rightarrow \mathfrak{N}$  such that  $Z = F(N)$   $\mathbf{P}$ -a.s. Furthermore, the map  $F$  is  $\mathcal{N}_t / \mathcal{Z}_{y^*(z,t)}$  measurable, i.e. for any  $t \geq 0$ ,*

$$\mathcal{Z}_{y^*(z,t)} = \sigma \{F(N(s)), s \leq t\}.$$

*Proof.* Let  $Y = (Z, N)$  and  $Y' = (Z', N')$  two solutions of  $g$ - $H_y$ , which are possibly defined on different probability spaces, and  $\mu$  and  $\mu'$  their respective distribution on  $(\mathfrak{N}^2; \mathcal{N}_\infty^{\otimes(2)})$ . On  $(\mathfrak{N}^3, \mathcal{N}_\infty^{\otimes(3)})$ , we define the probability measure

$$\nu(d\eta_1, d\eta_2, d\eta_3) = \mu_{\eta_3}(d\eta_1) \mu'_{\eta_3}(d\eta_2) d\pi(\eta_3).$$

We are going to prove that  $\eta_3$  is a  $\nu$  Poisson process. For, for  $i = 1, 2, 3$ , let  $F_i$  a bounded function,  $\mathcal{N}_{y^*(\eta_i, s)}^i / \mathcal{B}(\mathbf{R})$  measurable for  $i = 1, 2$  and  $\mathcal{N}_s^3 / \mathcal{B}(\mathbf{R})$  measurable for  $i = 3$ . We have

$$\begin{aligned} & \int_{\mathfrak{N}^3} F_1(\eta_1) F_2(\eta_2) F_3(\eta_3) (\tilde{\eta}_3(t) - \tilde{\eta}_3(s)) \nu(d\eta_1, d\eta_2, d\eta_3) \\ &= \int_{\mathfrak{N}} \left( \int_{\mathfrak{N}} F_1(\eta_1) \mu_{\eta_3}(d\eta_1) \right) \left( \int_{\mathfrak{N}} F_2(\eta_2) \mu'_{\eta_3}(d\eta_2) \right) (\tilde{\eta}_3(t) - \tilde{\eta}_3(s)) d\pi(\eta_3). \quad (51) \end{aligned}$$

where  $\tilde{\eta}_3(t) = \eta_3(t) - t$ . According to Lemma 6.3, the random variables

$$\int_{\mathfrak{N}} F_1(\eta_1) \mu_{\eta_3}(d\eta_1) \text{ and } \int_{\mathfrak{N}} F_2(\eta_2) \mu'_{\eta_3}(d\eta_2)$$

are  $\mathcal{N}_s^3$  measurable hence  $\mathcal{N}_s^{\otimes(3)}$ -measurable. It follows that the right-hand-side of (51) is equal to zero and that  $\eta_3$  is a  $(\nu, (\mathcal{N}_{y^*(\eta_1,t)}^1 \otimes \mathcal{N}_{y^*(\eta_2,t)}^2 \otimes \mathcal{N}_t^3, t \geq 0))$  unit rate Poisson process.

We thus have two solutions  $(\eta_1, \eta_3)$  and  $(\eta_2, \eta_3)$  on the same probability space. The path-wise uniqueness then implies that  $\eta_1 = \eta_2$ ,  $\nu$ -a.s. Thus,  $\mu(d\eta_1, d\eta_3) = \mu'(d\eta_2, d\eta_3)$  and the uniqueness in law holds.

In view of Lemma 2.5 and Corollary 4.5, the Doob Lemma says that there exist two functions  $F_i, i = 1, 2$  respectively measurable from  $(\mathfrak{N}, \mathcal{N}_\infty^3)$  to  $(\mathfrak{N}, \mathcal{N}_{y^*(\eta_i, \infty)}^i) = (\mathfrak{N}, \mathcal{N}_\infty^i), i = 1, 2$  such that  $\eta_1^{y^*(\eta_1, t)} = F_1(\eta_3^t)$  and  $\eta_2^{y^*(\eta_2, t)} = F_2(\eta_3^t)$  for any  $t \geq 0$ . Furthermore, for  $\pi$ -almost all  $\eta_3$ ,

$$\mu_{\eta_3} \otimes \mu_{\eta_3}(\eta_1 = \eta_2) = 1$$

and this implies that  $F_1 = F_2$ , i.e. there exists  $F$  such that  $\eta_1 = \eta_2 = F(\eta_3)$ .  $\square$

**Theorem 6.5** *Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space and  $N$  a unit Poisson process such that  $\mathcal{F} = \mathcal{N}$ . We denote by  $\pi$  the law of  $N$ . For any  $\dot{y} \in \mathcal{P}_2^{++}(\mathcal{N}, \pi_y)$ , there exists a (weak) solution to  $g\text{-}H_y$ .*

*Proof.* In view of Theorem 2.2,  $\pi_y$  is locally absolutely continuous with respect to  $\pi$  and the quasi-invariance theorem says that the process  $Y$  defined by

$$Y(t) := N(y^*(N, t)),$$

is a  $(\pi_y, \mathcal{F}^{Y^*})$  unit Poisson process. Thus, we have

$$N(t) = Y(y(N, t))$$

and  $y(N, t)$  is a  $\mathcal{F}^{Y^*}$ -stopping time, hence  $(N, Y, \mathcal{F}^{Y^*})$  solves  $g\text{-}H_y^w$ .  $\square$

**Corollary 6.6** *Consider  $(\Omega, \mathcal{F}, \mathbf{P})$  be a filtered probability space and  $N$  a unit Poisson process. Let  $\dot{y} \in \mathcal{P}_2^{++}(\mathcal{N}, \pi_y)$ . Then, there exists a unique strong solution to  $g\text{-}H_y$ .*

*Proof.* The weak existence is guaranteed by Theorem 6.5. Strong uniqueness follows from Theorem 6.1. We conclude with Theorem 6.4.  $\square$

We can now retrieve the classical existence result for classical Hawkes processes [16, 22].

**Corollary 6.7** *Let  $\dot{y}$  be a predictable process which satisfies the following hypothesis*

$$\dot{y}(t) = \varphi \left( \alpha + \int_0^t h(t-s) dN(s) \right)$$

where  $\alpha > 0$ ,  $\varphi$  is a Lebesgue a.s. positive Lipschitz function and  $h$  is a non-negative measurable function such that

$$\|\varphi\|_{\text{Lip}} \int_0^\infty h(t) dt < 1. \quad (52)$$

Then, there exists a unique strong solution to  $g\text{-}H_y$ .

*Proof.* Let  $T > 0$  and

$$\dot{y}_T(t) = \begin{cases} \dot{y}(s) & \text{if } s \leq T \\ 1 & \text{if } s > T. \end{cases}$$

We first prove that there exists a solution to  $\mathfrak{H}_{y_T}$ . We have to prove that

$$\pi_y \left( \int_0^T \left(1 - \sqrt{\dot{y}(s)}\right)^2 ds < \infty \right) = 1. \quad (53)$$

It is easily seen that

$$\left(\sqrt{1+x} - 1\right)^2 \leq x^2 \mathbf{1}_{\{|x| \leq 1\}} + x \mathbf{1}_{\{|x| > 1\}} \quad \forall x \geq -1.$$

Since  $\varphi$  is supposed to be Lipschitz continuous, we get

$$\begin{aligned} \int_0^T \left(1 - \sqrt{\dot{y}(s)}\right)^2 ds &\leq T + \int_0^T |\dot{y}(s) - 1| ds \\ &\leq (1 + |\varphi(0) - 1| + \alpha \|\varphi\|_{\text{Lip}})T + \|\varphi\|_{\text{Lip}} \int_0^T \int_0^s h(s-u) dN(u) ds. \end{aligned}$$

Thus, proving (53) amounts to prove that

$$\pi_y \left( \int_0^T \int_0^s h(s-u) dN(u) ds < \infty \right) = 1.$$

Denote by  $H$  the first quadrature of  $h$ :

$$H(t) = \int_0^t h(s) ds.$$

By Fubini's Theorem, we have

$$\int_0^T \int_0^s h(s-u) dN(u) ds = \int_0^T H(T-s) dN(s) \leq H(T)N(T).$$

The same argument shows that

$$\mathbf{E}_{\pi_y} [N(T)] \leq cT + \|\varphi\|_{\text{Lip}} H(T) \mathbf{E}_{\pi_y} [N(T)].$$

In view of (52), this induces that  $\mathbf{E}_{\pi_y} [N(T)]$  is finite and thus that (53) holds. Then, Theorem 6.6 ensures the existence of the Hawkes process associated to  $y$  on  $[0, T]$  for any  $T > 0$ . We denote this process by  $Y_T$ . Since  $y$  is predictable, for all  $r \leq T \leq S$ ,

$$\begin{aligned} Y_T(r) &= N(y_T(Y_T, r)) \\ &= N(y_S(Y_T, r)). \end{aligned}$$

By uniqueness, this means that  $Y_T$  and  $Y_S$  coincide on  $[0, T]$ . We can then define

$$Y(r) = Y_r(r),$$

which satisfies

$$Y(r) = N(y_r(Y(r), r)) = N(y(Y(r), r))$$

by the very definition of  $y_r$ . We have thus proved the existence of a strong solution to the Hawkes problem associated to  $y$ . The uniqueness comes from Theorem 6.1.  $\square$

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