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Laure Coutin, Laurent Decreusefond

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Invertibility of functionals of the Poisson process and applications

L. Coutin

L. Decreusefond

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We show that solving SDEs with constant volatility on the Wiener space is the analog of constructing Hawkes-like processes, i.e. self excited point process, on the Poisson space. Actually, both problems are linked to the invertibility of some transformations on the sample paths which respect absolute continuity: adding an adapted drift for the Wiener space, making a random time change for the Poisson space. Following previous investigations by Üstünel on the Wiener space, we establish an entropic criterion on the Poisson space which ensures the invertibility of such a transformation. As a consequence of this criterion, we improve the variational representation of the entropy with respect to the Poisson process distribution. Pursuing the Wiener-Poisson analogy so established, we define several notions of generalized Hawkes processes as weak or strong solutions of some fixed point equations and show a Yamada-Watanabe like theorem for these new equations. As a consequence, we find another construction of the classical (even non linear) Hawkes processes without the recourse to a Poisson measure.

Keywords: Girsanov Theorem, Hawkes processes, Invertibility, Random Time Change

Math Subject Classification: 60G55

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1 Introduction

The simplest non linear stochastic differential equations are those of the form:

$$X(0) = 0 \text{ and } dX(t) = \dot{u}(X(t)) dt + dB(t) \iff X(t) = \int_0^t \dot{u}(X(s)) ds + B(t), \quad (1)$$

where B is a standard Brownian motion. We call these equations where the coefficient in front of B is equal to 1, *volatility-1* Brownian SDEs. Denote by W the space of continuous functions over $[0, 1]$, null at time 0, equipped with the Wiener measure μ . We may view the process X as a map from W into itself by the construction:

$$\begin{aligned} X : W &\longrightarrow W \\ \omega &\longmapsto \left(t \mapsto X(\omega, t) \right). \end{aligned}$$

Consider also the map:

$$\begin{aligned} U : W &\longrightarrow W \\ \omega &\longmapsto \left(t \mapsto \omega(t) - \int_0^t \dot{u}(\omega(s)) ds \right). \end{aligned} \quad (2)$$

Then, (1) is equivalent to say that X is a solution of the equation

$$U \circ X = \text{Id}_W, \mu\text{-a.s.}$$

Hence, to solve (1) is to invert U . In [24] and further on in [12, 18, 25], the authors showed that U is invertible if and only if

$$H(U^\# \mu | \mu) = \frac{1}{2} \mathbf{E} \left[\int_0^1 \dot{u}(\omega(s))^2 ds \right] \quad (3)$$

where $H(U^\# \mu | \mu)$ is the relative entropy between $U^\# \mu$, the image measure of μ by U , and μ itself.

Let \mathbb{D} be the Skorohod space of right continuous with left limits (rcll for short) functions equipped with the Poisson measure π , i.e. the law of a unit rate Poisson process on \mathbf{R}^+ . The initial interpretation of the Poisson counterpart of (1) is to solve

$$Y(t) = \int_0^t \dot{u}(Y(s)) ds + N(t), \quad (4)$$

as the inversion of the map \tilde{U} , defined formally as U but from \mathbb{D} into itself. That is to say that we seek a rcll process Y which satisfies $\tilde{U} \circ Y = \text{Id}_{\mathbb{D}}$, π -a.s. This appears to be inconsistent with the methods of the previously mentioned works. Actually, to grasp the difference between the Wiener and Poisson settings and consequently to appreciate the inherent differences between equations (1) and (4), we must go back to the basics of the Girsanov theorem. A measure ν on W is absolutely continuous with respect to μ if there exists an adapted process u in the Cameron-Martin space such $B - u$ is a ν -local martingale of square bracket ($t \mapsto t$).

Since the square bracket of a semi-martingale is unchanged by the addition of a finite variation process, the Lévy's characterization theorem says that $B - u$ is a ν Brownian motion. It is this quasi-invariance which is used to prove that SDEs like (1), have weak solutions under mild assumptions on \dot{u} and which is the key to the investigations about invertibility.

Now, the main obstacle to a direct generalisation of this approach to the Poisson space is that the Girsanov theorem for the Poisson process is best expressed in terms of point processes rather than in terms of rcl functions as it involves the notion of compensator which is specific to the former. Let \mathfrak{N} be the space of configurations on $[0, +\infty)$ (see the exact definition below) and π the probability on \mathfrak{N} such that the canonical process N is a Poisson process of unit intensity. The Girsanov theorem says that if ν is absolutely continuous with respect to π , there exists a non-decreasing predictable process, denoted by y , such that $N - y$ is a ν local martingale. The difference here is that the ν -compensator of N is y so neither N nor $N - y$ (which is not even a point process) are Poisson processes under ν . Otherwise stated, the transformation of sample paths induced by an absolutely continuous change of probability in the Poisson framework is no longer a translation, i.e. the addition to the nominal path of a regular function.

We have thus to construct another transformation of the sample paths of N such that the process we obtain, after a change of probability measure, is still a ν -Poisson process of unit intensity. This question has been seldom addressed (see [4, 8]) and not in a form which is convenient for our present goal. It turns out that it is the process $N(y^*(t))$, where y^* is the right-inverse of y , which plays the role of $B - u$ in the Poisson space. Thus, the true analog of the map U defined in (2) is the map \mathbf{Y} defined as

$$\begin{aligned} \mathbf{Y} : \mathfrak{N} &\longrightarrow \mathfrak{N} \\ N &\longmapsto Y := N \circ y^* \end{aligned}$$

and not the map \tilde{U} . Consequently, the true analog of the invertibility of U is to invert \mathbf{Y} in the space of configurations. We show in Lemma 3.2 that this amounts to find a point process Z with compensator z such that

$$z^*(N, y^*(Z, t)) = t, \text{ for any } t \geq 0. \quad (5)$$

The interpretation of this equation is the following. Given a point process Y on the half-line which is adapted to a filtration \mathcal{F} , we can always construct a \mathcal{F} -predictable process y , known as its compensator, such that $Y - y$ is a \mathcal{F} -local martingale (see [14]). The reverse question which is, given a \mathcal{F} -predictable, non-decreasing, right-continuous, null at time 0, process y , to devise the existence of a point process Y such that y is the \mathcal{F}_Y -compensator of Y has, to the best of our knowledge, never been addressed in full generality. The only situation we are aware of, where we only have a partial solution, is related to the notion of Hawkes processes. Recall that, given two deterministic functions ϕ and ψ , a Hawkes process [13, 20] is a point process H such that $H(t) - \int_0^t \psi \left(\lambda + \int_0^s \phi(s-r) dH(r) \right) ds$ is a local martingale. The usual way to proceed is to construct H as the solution of a differential equation driven by a marked Poisson process Φ on $\mathbf{R}^+ \times \mathbf{R}^+$. This means that H and its compensator

$$y(t) = \int_0^t \psi \left(\lambda + \int_0^s \phi(s-r) dH(r) \right) ds \quad (6)$$

are adapted with respect to the σ -field generated by Φ and not to the minimal σ -field we could hope for, which is the one generated by H itself. We show below that, if N is a unit rate Poisson process of $[0, +\infty)$, the map \mathbf{Y} is right invertible (i.e. (5) is satisfied for some point process Z) if the point process Z is the solution of the equation

$$Z(t) = N\left(y(Z, t)\right), \quad (7)$$

see Theorem 3.4 for a precise statement. The form of (7) entails that $y(Z, t)$ is the compensator of Z and Corollary 3.5 ensures that it is adapted to the minimal filtration generated by Z . The process Z is what we call a generalized Hawkes process (g-Hawkes for short) as our results do not depend on a particular expression of y as in (6). This means that constructing a g-Hawkes process is the Poissonian analogue to solving volatility-1 Brownian SDEs, like (1), in the Wiener space. This point of view, which, to the best of our knowledge, is new, has major consequences as we can now transfer the problems known on SDEs (weak, strong and martingales solutions, perturbations, stationarity, etc.) to g-Hawkes processes. We only focus here on the different notions of solutions for the g-Hawkes problem and show the analogue of the Yamada-Watanabe theorem (see Theorem 5.3). We can then construct a classical, possibly with a non linear intensity, Hawkes process without the recourse to a larger probability space as usually done ([10, 20, 13] and references therein).

The variational representation of entropy is a crucial theorem of the theory of large deviations [11], also known as the Boué-Dupuis or Borell formula [5, 6] for the Gaussian measure on \mathbf{R}^n . It has been extended to the Wiener space by Léhec in [19] and with weaker hypothesis by Üstünel in [24], as a consequence of the entropic criterion for the invertibility of U , see (3). The core of the proof involves inverting a map akin to U . Without the entropic criterion, the only known alternative approach is to solve equation (1), which imposes certain restrictions on the scope of \dot{u} that can be considered. With the entropic criterion, however, it is no longer necessary to solve an SDE; instead, one can pass to the limit in equation (3), thereby expanding the space of drifts that can be considered (see also [12]).

In this work, we follow a similar methodology to establish an analogous formula for the Poisson space (see Theorem 4.2), based on the Poisson entropic criterion we establish (Theorem 3.9): \mathbf{Y} is left invertible if and only if

$$H(\mathbf{Y}^\# \pi | \pi) = \mathbf{E}_\pi \left[\int_0^\infty \left(\dot{y}^*(s) \log \dot{y}^*(s) - \dot{y}^*(s) + 1 \right) ds \right], \quad (8)$$

where \dot{y}^* is the derivative of the reciprocal function of y . A similar result has been previously established in [26], but it requires to embed the Poisson process within the broader probability space of marked point processes.

This paper is organized as follows: In Section 2, we define what we consider here as random time changes and describe their action on stochastic integrals. We establish the quasi-invariance theorem and compute the Radon-Nikodym density of the push-forward of the Poisson measure (see below for the definition) by a random time change. In Section 3, we combine these results to obtain the entropic criterion which guarantees the invertibility of a map like \mathbf{Y} . We take profit of the robustness of the entropic criterion with respect to limit procedure to obtain a variational representation of the entropy in Section 4. The notion of

weak, strong and martingale g-Hawkes problem are introduced in Section 5. We then prove the Yamada-Watanabe theorem for the g-Hawkes problem.

2 Changes of time and changes of measures

2.1 Preliminaries

Definition 2.1 Let \mathfrak{N} be the set of locally finite, simple configurations on $E = (0, +\infty)$ equipped with the vague convergence. We denote by ω its generic element. For each $\omega \in \mathfrak{N}$, there exist $(T_n(\omega), n \geq 1)$ such that $T_n(\omega) < T_{n+1}(\omega)$, $T_n(\omega)$ tends to infinity as n tends to infinity and

$$\omega = \sum_{n \geq 1} \epsilon_{T_n(\omega)}$$

where ϵ_a is the Dirac mass at a . We denote by N , the counting process associated to this measure by:

$$N(\omega, t) = \omega([0, t]).$$

Note that given the sample path of N , we can retrieve ω as

$$T_n(\omega) = \inf\{t, N(t) = n\} \text{ or } (T_n(\omega), n \geq 1) = (t, \Delta N(t) = 1)$$

where $\Delta N(t) = N(t) - N(t^-)$. This means that we can identify a sample path of N with an element of \mathfrak{N} . A point process is a random variable with values in \mathfrak{N} .

The filtrations do play an important role in the following. We denote by $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ a generic right continuous filtration on \mathfrak{N} . We denote by \mathcal{F}_∞ the whole σ -field, i.e. $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. The minimal filtration under which the canonical process N is measurable is

$$\mathcal{N}_t = \sigma\{N(s), s \leq t\}. \quad (9)$$

We follow the presentation of [14] where the notion of predictability is defined without any reference to the completion of the filtration.

Definition 2.2 For any filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ on \mathfrak{N} , a real-valued process $(X(t), t \geq 0)$ is called \mathcal{F} -predictable and belongs to $\mathcal{P}(\mathcal{F})$, if the application $(\omega, t) \mapsto X(\omega, t)$ is measurable with respect to the σ -algebra \mathcal{P} on $\mathfrak{N} \times [0, +\infty)$ generated by the applications $(\omega, t) \mapsto Y(\omega, t)$ which are \mathcal{F}_t measurable in ω and left-continuous in t . We denote by $\mathcal{P}(\mathcal{F})$ the set of processes which are \mathcal{F} -predictable.

Theorem 2.1 *Let \mathcal{F} be a filtration on \mathfrak{N} and μ a probability measure on $(\mathfrak{N}, \mathcal{F}_\infty)$. The next result comes from proposition (3.40) and theorem (3.42) of [15].*

1. *Then, there exists a unique predictable (up to μ -null set) process y which is non-decreasing, right continuous, null at time 0 and such that for any $q \geq 1$,*

$$t \mapsto N(t \wedge T_q(N)) - y(t \wedge T_q(N))$$

is a uniformly integrable $(\mathfrak{N}, \mathcal{F}, \mu)$ -martingale.

2. For the converse, we need to assume that the filtration is the minimum filtration under which the canonical process is measurable. For any process $y \in \mathcal{P}(\mathcal{N})$, non-decreasing, null at time 0 and right-continuous, there exists a unique probability measure on $(\mathfrak{N}, \mathcal{N}_\infty)$, denoted by π_y , such that $N - y$ is a $(\mathcal{N}_t, t \geq 0)$ local martingale and $(T_q(N), q \geq 1)$ is a localizing sequence.

When $y = \text{Id}$, i.e., when the canonical process is a unit rate Poisson process, we prefer to use the notation π instead of π_{Id} .

2.2 Absolute continuity and equivalence

We summarize the results on local absolute continuity on the Poisson space which can be found in theorems 8.32 and 8.35 and corollary 8.37 of [15].

Theorem 2.2 Consider a filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ on \mathfrak{N} . Let $\dot{\kappa}$ be a non-negative, locally integrable \mathcal{F} -predictable process. We set

$$\kappa(t) = \int_0^t \dot{\kappa}(s) \, ds.$$

We consider π_κ as defined in Theorem 2.1.

1. If a probability measure ν on \mathfrak{N} is locally absolutely continuous with respect to π_κ along \mathcal{F} (that is denoted $\nu \ll_{\text{loc}, \mathcal{F}} \pi_\kappa$) then there exists a unique process \dot{y}_κ which is non-negative and \mathcal{F} -predictable such that

$$\forall t \geq 0, \nu \left(\int_0^t \left(1 - \sqrt{\dot{y}_\kappa(s)} \right)^2 \dot{\kappa}(s) \, ds < \infty \right) = 1 \quad (10)$$

and

$$t \mapsto N(t) - \int_0^t \dot{y}_\kappa(s) \dot{\kappa}(s) \, ds \text{ is a } (\mathcal{F}, \nu)\text{-local martingale.} \quad (11)$$

We set

$$y_\kappa(t) = \int_0^t \dot{y}_\kappa(s) \dot{\kappa}(s) \, ds$$

which belongs to $\mathcal{P}(\mathcal{F})$, is non decreasing, right continuous and null at time 0. Hence, with the notations introduced above, $\nu = \pi_{y_\kappa}$.

2. If π_{y_κ} is absolutely continuous with respect to π_κ on \mathcal{F}_∞ (that is denoted by $\pi_{y_\kappa} \ll \pi_\kappa$) then

$$\pi_{y_\kappa} \left(\int_0^\infty \left(1 - \sqrt{\dot{y}_\kappa(s)} \right)^2 \dot{\kappa}(s) \, ds < \infty \right) = 1. \quad (12)$$

For the converse part, we need to assume that $\mathcal{F} = \mathcal{N}$, the minimal filtration which makes N adapted. In this situation, consider \dot{y} a locally integrable, non negative and

\mathcal{N} -predictable process and π_{y_κ} the probability measure on \mathfrak{N} , which satisfies (10) and (11).

3. Then, π_{y_κ} is locally absolutely continuous with respect to π_κ along \mathcal{N} and

$$\begin{aligned} \Lambda_{y_\kappa}(N, t) &:= \frac{d\pi_{y_\kappa}}{d\pi_\kappa} \Big|_{\mathcal{N}_t} \\ &= \begin{cases} \exp\left(\int_0^t \log \dot{y}_\kappa(s) dN(s) + \int_0^t (1 - \dot{y}_\kappa(s)) \dot{\kappa}(s) ds\right) & \text{if } t \leq S_m, \\ 0 & \text{if } t \geq \limsup_m S_m, \end{cases} \end{aligned}$$

where for any integer $m \geq 1$,

$$S_m = \inf \left\{ t, \int_0^t \left(1 - \sqrt{\dot{y}_\kappa(s)}\right)^2 \dot{\kappa}(s) ds \geq m \right\}.$$

4. Finally, the probability measure π_{y_κ} is absolutely continuous with respect to π_κ on $(\mathfrak{N}, \mathcal{N}_\infty)$ if and only if (11) and (12) are satisfied.

The next result gives some necessary and sufficient conditions for the equivalence of π_y and π_κ , see Proposition (7.11) of [15].

Lemma 2.3 Assume that ν is absolutely continuous with respect to μ on \mathcal{N}_∞ and set

$$\Lambda(t) = \frac{d\nu}{d\mu} \Big|_{\mathcal{N}_t}. \quad (13)$$

Then ν and μ are equivalent if and only the following two conditions are satisfied:

i) The local martingale $(\Lambda(t), t \geq 0)$ is uniformly integrable, i.e. there exists $\Lambda \in L^1(\mu)$ such that

$$\Lambda(t) = \mathbf{E}_\mu[\Lambda | \mathcal{N}_t].$$

ii) The random variable Λ is positive μ -a.s.

In view of this theorem, we introduce the following sets of processes.

Definition 2.3 Let \mathcal{F} be a filtration on $\mathfrak{N} \times \mathbf{R}^+$ and $\dot{\kappa}$ a non-negative predictable process. Consider the probability measure π_κ on \mathfrak{N} and let $\mathcal{D}^{++}(\mathcal{F}, \pi_\kappa)$ (respectively $\mathcal{D}^+(\mathcal{F})$) be the set of positive (respectively non-negative) \mathcal{F} -predictable processes \dot{y} such that for any $t \geq 0$

$$\pi_\kappa(y_\kappa(t) < +\infty) = 1 \text{ and } \pi_\kappa\left(\lim_{t \rightarrow \infty} y_\kappa(t) = +\infty\right) = 1,$$

where the process y is defined by

$$y_\kappa(\omega, t) = \int_0^t \dot{y}(\omega, s) \dot{\kappa}(s) ds.$$

We also introduce the following subset of $\mathcal{P}^{++}(\mathcal{F}, \pi_\kappa)$:

$$\mathcal{P}_2^{++}(\mathcal{F}, \pi_\kappa) = \left\{ \dot{y} \in \mathcal{P}^{++}(\mathcal{F}, \pi_\kappa), \pi_\kappa \left(\int_0^\infty (1 - \sqrt{\dot{y}(s)})^2 \dot{\kappa}(s) ds < \infty \right) = 1 \right\}.$$

The most restricted class we consider is $\mathcal{P}_\infty^{++}(\mathcal{F}, \pi_\kappa)$ of processes $\dot{y} \in \mathcal{P}^{++}(\mathcal{F}, \pi_\kappa)$ for which there exist $\varepsilon \in (0, 1)$ and $T > 0$ such that

$$\varepsilon \leq \dot{y}(s) \leq \frac{1}{\varepsilon}, \quad \forall s \geq 0 \text{ and } \dot{y}(s) = 1 \text{ for } s \geq T, \quad \pi_\kappa - \text{a.s.}$$

Note that if \dot{y} belongs to $\mathcal{P}_\infty^{++}(\mathcal{F}, \pi_\kappa)$ then $(\Lambda_{y_\kappa}(N, t), t \geq 0)$ is uniformly integrable.

As long as there is a finite number of jumps of the process N , which happens on any finite interval, the argument of the exponential in Λ_{y_κ} cannot be minus infinity hence Λ_{y_κ} is positive. We give a sufficient condition which ensures that this may not happen even on the half line.

Lemma 2.4 *If the predictable process $(\dot{y} - 1)$ belongs to $L^1(\mathfrak{N} \times \mathbf{R}^+, \pi_\kappa \otimes \dot{\kappa}(s) ds)$ then, π_κ -a.s.,*

$$\begin{aligned} \Lambda_{y_\kappa} &:= \lim_{t \rightarrow \infty} \Lambda_{y_\kappa}(t) \\ &= \exp \left(\int_0^\infty \log \dot{y}(s) dN(s) + \int_0^\infty (1 - \dot{y}(s)) \dot{\kappa}(s) ds \right) \in (0, +\infty). \end{aligned} \quad (14)$$

Proof. Since, for any $x \geq -1$,

$$(1 - \sqrt{1+x})^2 \leq |x|,$$

we have

$$\mathbf{E}_{\pi_\kappa} \left[\int_0^\infty (1 - \sqrt{\dot{y}(s)})^2 \dot{\kappa}(s) ds \right] \leq \mathbf{E}_{\pi_\kappa} \left[\int_0^\infty |\dot{y}(s) - 1| \dot{\kappa}(s) ds \right] < \infty.$$

Hence, for π_κ almost all ω , there exists $m(\omega) < \infty$ such that $S_m(\omega) = \infty$ for any $m \geq m(\omega)$ (S_m is defined in Theorem 2.2 point 3) and then we have

$$\begin{aligned} \Lambda_{y_\kappa} &= \lim_{t \rightarrow \infty} \Lambda_{y_\kappa}(t) \\ &= \left(\prod_{n=1}^\infty \dot{y}(T_n) \right) \exp \left(\int_0^\infty (1 - \dot{y}(s)) \dot{\kappa}(s) ds \right). \end{aligned}$$

The infinite product is convergent as soon as

$$\sum_{n=1}^\infty |\dot{y}(T_n) - 1| < \infty.$$

Since

$$\mathbf{E}_{\pi_\kappa} \left[\sum_{n=1}^\infty |\dot{y}(T_n) - 1| \right] = \mathbf{E}_{\pi_\kappa} \left[\int_0^\infty |\dot{y}(s) - 1| \dot{\kappa}(s) ds \right],$$

we see that $(\dot{y} - 1) \in L^1(\mathfrak{N} \times \mathbf{R}^+, \pi_\kappa \otimes \dot{\kappa}(s) ds)$ entails (14). □

2.3 Random time changes

The sequel of this paper will be based on the notion of random time change. We refer to [15, chapter 10] or [3] for details on this notion and its links with stochastic calculus.

Definition 2.4 On $(\mathfrak{N}, \mathcal{F} = (\mathcal{F}_t, t \geq 0))$, a random change of time is a right continuous, null at time 0, non-decreasing process $(\eta(t), t \geq 0)$ such that

$$\eta(t) < \infty, \forall t \geq 0, \lim_{t \rightarrow \infty} \eta(t) = +\infty$$

and for any $t \geq 0$, $\eta(t)$ is an \mathcal{F} -stopping time. We denote by \mathcal{F}^η the filtration $(\mathcal{F}_{\eta(t)}, t \geq 0)$. For X a process, $\tau_\eta(X)$ is the process defined by

$$\tau_\eta(X)(t) = X(\eta(t)).$$

Note that many changes of time are given by the right inverse of non decreasing predictable processes: For y such a process,

$$y^*(\omega, t) = \inf\{s, y(\omega, s) > t\}$$

is a change of time (with the usual convention that the infimum of the empty set is infinite). According to [15, Chapter 10],

$$(y(\omega, t) < s) = (y^*(\omega, s^-) > t) \quad (15)$$

$$(y^*(\omega, t) < s) = (y(\omega, s^-) > t). \quad (16)$$

If y is continuous, we have

$$y(\omega, y^*(\omega, t)) = t \text{ and } y^*(\omega, y(\omega, t)) = \inf\{u, y^*(\omega, u) > y^*(\omega, t)\}.$$

Note that if y belongs to $\mathcal{P}^{++}(\mathcal{F}, \mu)$, then y is almost-surely an homeomorphism from \mathbf{R}^+ onto itself, hence

$$y(\omega, y^*(\omega, t)) = y^*(\omega, y(\omega, t)) = t \text{ and } \tau_{y^*}(X)(t) = X(y^*(t)), \text{ for all } t \geq 0, \mu\text{-a.s.} \quad (17)$$

Furthermore, (15) entails that $y(\omega, t)$ is an \mathcal{F}^{y^*} stopping time.

In the Brownian setting, the entropic criterion involves the L^2 norm of the drift. The square function is here replaced by a convex function which appears frequently in Poissonian settings (see for instance [2] and references therein).

Definition 2.5 Consider m , the smooth, convex, non-negative function defined on $[-1, \infty)$ by

$$m(x) = \begin{cases} (x+1) \log(x+1) - x & \text{if } x > -1, \\ 1 & \text{if } x = -1. \end{cases}$$

and L_m , the corresponding Orlicz space [1]:

$$L_m = \left\{ f : \mathbf{R}^+ \rightarrow [-1, \infty), \int_0^\infty m(f(s)) \, ds < \infty \right\}.$$

We view the time change as a map from \mathfrak{N} into itself.

Definition 2.6 For $\dot{y} \in \mathcal{D}^{++}(\mathcal{N})$, let

$$\begin{aligned} \tau_{y^*} : \mathfrak{N} &\longrightarrow \mathfrak{N} \\ N &\longmapsto \tau_{y^*}(N) : \left(t \mapsto N(y^*(N, t)) \right). \end{aligned} \quad (18)$$

We denote by Y the process $\tau_{y^*}(N)$.

Note that the process Y is adapted to the filtration

$$\mathcal{F}^{y^*} = (\mathcal{F}_{y^*(t)}, t \geq 0)$$

and has (\mathcal{F}^{y^*}, π) -compensator y^* (see [15, Theorem 10.17]).

Definition 2.7 For a probability measure μ on \mathfrak{N} , we denote by $\tau_{y^*}^\# \mu$ the distribution of the process Y on \mathfrak{N} or equivalently the push-forward of the measure μ by the map τ_{y^*} .

One way to understand this transformation is to consider that the locations of the atoms of the underlying configuration N are fixed once and for all. They are discovered at unit rate with N and at random speed by Y : The k -th jump of Y occurs at time t if and only $y^*(N, t) = T_k(N)$ hence

$$y^*(N, T_k(Y)) = T_k(N). \quad (19)$$

Lemma 2.5 For $\dot{y} \in \mathcal{D}^{++}(\mathcal{N})$, let $\mathcal{Y}_t = \sigma(Y(u), u \leq t)$. Then, we have

$$\mathcal{Y}_t \vee \sigma(y^*(N, s), s \leq t) = \mathcal{N}_{y^*(N, t)}.$$

Furthermore, $\mathcal{Y}_\infty \subset \mathcal{N}_\infty^{y^*} = \mathcal{N}_\infty$.

Proof. For any $s \leq t$,

$$y^*(N, s) \in \mathcal{N}_{y^*(N, s)} \subset \mathcal{N}_{y^*(N, t)}.$$

The definition of Y induces that $Y(t)$ is $\mathcal{N}_{y^*(t)}$ measurable hence

$$\mathcal{Y}_t \subset \mathcal{N}_{y^*(N, t)}.$$

It follows that

$$\mathcal{Y}_t \vee \sigma(y^*(N, s), s \leq t) \subset \mathcal{N}_{y^*(N, t)}.$$

Conversely, let $A \in \mathcal{N}_{y^*(N, t)}$, according to [15, Proposition 3.40],

$$\mathbf{1}_A = \sum_{q=0}^{\infty} \Psi_q \left(T_1(N), \dots, T_q(N) \right) \mathbf{1}_{\{T_q(N) \leq y^*(N, t) < T_{q+1}(N)\}},$$

where Ψ_q is measurable from $(\mathbf{R}^+)^q$ to $\{0, 1\}$. It follows from the definition of Y (see (19)) that

$$\mathbf{1}_A = \sum_{q=0}^{\infty} \Psi_q \left(y^*(N, T_k(Y)), k = 1, \dots, q \right) \mathbf{1}_{\{Y(t)=q\}}.$$

Since y^* is continuous, for $u \leq t$, we have

$$y^*(N, u) = \lim_{n \rightarrow \infty} \sum_{i=0}^{2^n-1} y^*(N, \frac{i}{2^n}) \mathbf{1}_{[\frac{i-1}{2^n}, \frac{i}{2^n})}(u),$$

thus, for $k \leq q$,

$$y^*(N, T_k(Y)) \mathbf{1}_{\{Y(t)=q\}} \in \mathcal{Y}_t \wedge \sigma(y^*(s), s \leq t).$$

Hence the converse embedding also holds.

The inclusion $\mathcal{Y}_\infty \subset \mathcal{N}_\infty^{y^*}$ is immediate from the first part of the proof. The equality between $\mathcal{N}_\infty^{y^*}$ and \mathcal{N}_∞ comes from [15, page 326]. \square

The following theorem is Theorem (10.19) of [15] and Lemma 1.3 of [3].

Theorem 2.6 *Let $\dot{y} \in \mathcal{P}^{++}(\mathcal{F})$. For $r \in \mathcal{P}(\mathcal{F})$ such that its stochastic integral is well defined, we have*

$$\tau_{y^*} \left(\int_0^\cdot r(N, s) ds \right) = \int_0^\cdot r(N, y^*(N, s)) dy^*(s) \quad (20)$$

and for the stochastic integrals,

$$\tau_{y^*} \left(\int_0^\cdot r(N, s) dN(s) \right) = \int_0^\cdot r(N, y^*(N, s)) dY(s). \quad (21)$$

We can extend τ_{y^*} to stochastic processes which are measurable maps from $(\mathfrak{N} \times \mathbf{R}^+, \mathcal{N}_\infty \otimes \mathcal{B}(\mathbf{R}^+))$ to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$. For μ a probability measure on \mathfrak{N} , we denote by $L^0(\mathfrak{N} \times \mathbf{R}^+, \mathbf{R}; \mu)$ the set of stochastic processes equipped with the topology of convergence in probability. We denote by \mathbf{Y} this extension, which is not to be confused with the process Y . For the sake of simplicity, we henceforth use the notation \mathbf{Y} even for τ_{y^*} .

Definition 2.8 The map \mathbf{Y} is defined by

$$\begin{aligned} \mathbf{Y} : L^0(\mathfrak{N} \times \mathbf{R}^+, \mathbf{R}; \mu) &\longrightarrow L^0(\mathfrak{N} \times \mathbf{R}^+, \mathbf{R}; \mu) \\ U &\longmapsto \left((N, s) \mapsto U(Y, y^*(N, s)) \right). \end{aligned}$$

For instance, if we denote by N^a the process N stopped at time a :

$$N^a(t) = N(t \wedge a),$$

we have

$$\begin{aligned} Y^{y(a)}(t) &= Y(y(N, a) \wedge t) \\ &= N(y^*(N, y(N, a) \wedge t)) \\ &= N(a \wedge y^*(N, t)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} y^*(N^a)(t) &= N^a(y^*(N, t)) \\ &= N(y^*(N, t) \wedge a) \end{aligned}$$

so that

$$N^a \circ \mathbf{Y} = Y^{y(N, a)}. \quad (22)$$

We then have the following composition rules.

Theorem 2.7 *Let $\dot{y} \in \mathcal{P}^{++}(\mathcal{N})$. For $r \in \mathcal{P}(\mathcal{N})$ such that its stochastic integral with respect to Y is well defined, we have*

$$\left(\int_0^\cdot r(N, s) ds \right) \circ \mathbf{Y} = \int_0^\cdot r(Y, y^*(N, s)) dy^*(N, s) \quad (23)$$

and

$$\left(\int_0^\cdot r(N, s) dN(s) \right) \circ \mathbf{Y} = \int_0^\cdot r(Y, y^*(N, s)) dY(s). \quad (24)$$

■ **Remark 1** These formulas are no longer valid if \dot{y} is not supposed to be positive or if y is not an homeomorphism almost surely. ■

Proof. Since y defines a diffeomorphism from \mathbf{R}^+ onto itself, the change of variable $u = y(N, s)$ yields

$$\begin{aligned} \left(\int_0^\cdot r(N, s) ds \right) \circ \mathbf{Y}(t) &= \int_0^{y^*(N, t)} r(Y, s) ds \\ &= \int_0^t r(Y, y^*(N, u)) \dot{y}^*(N, u) du \end{aligned}$$

and (23) holds.

To prove (24), it is sufficient to prove it for simple predictable process, i.e. we assume that

$$r(N, s) = A(N) \mathbf{1}_{(a, b]}(s)$$

for some $A \in \mathcal{N}_a$. We have

$$\int_0^\cdot r(N, s) dN(s) = A(N)(N^b - N^a)$$

where N^a is the process N stopped at time a . On the one hand, according to the definition of \mathbf{Y} and to (22), we have

$$\left(A(N)(N^b - N^a) \right) \circ \mathbf{Y} = A(Y) \left(Y^{y(b)} - Y^{y(a)} \right).$$

On the other hand,

$$r(Y, y^*(N, s)) = A(Y) \mathbf{1}_{(y(N,a), y(N,b)]}(s)$$

hence

$$\int_0^\cdot r(Y, y^*(N, s)) dY(s) = A(Y) \left(Y^{y(N,b)} - Y^{y(N,a)} \right).$$

The proof is thus complete. \square

2.4 Quasi-invariance

The next result is the exact analog of the quasi-invariance theorem for the Wiener space (often quoted as the Girsanov theorem): find a perturbation of the sample paths and a change of probability which compensate each other. As mentioned above, for point processes, translations by an element of the Cameron-Martin space are replaced by time changes (see [8] and [4] for marked point processes). We first evaluate how a time change modifies the Radon-Nikodym derivative.

Lemma 2.8 *Let $\dot{\kappa}$ be a deterministic positive function from \mathbf{R}^+ into itself such that*

$$\kappa(t) = \int_0^t \dot{\kappa}(s) ds < \infty, \forall t \geq 0 \text{ and } \lim_{t \rightarrow \infty} \kappa(t) = +\infty.$$

Let ν be locally absolutely continuous with respect to π_κ along \mathcal{F} and let \dot{y}_κ denote its Girsanov factor, i.e. $\nu = \pi_{y_\kappa}$. Assume that \dot{y}_κ belongs to $\mathcal{D}_2^{++}(\mathcal{F}, \pi_{y_\kappa})$. Then, π_{y_κ} is locally absolutely continuous with respect to π_κ along $\mathcal{F}^{y_\kappa^}$ and we have*

$$\Lambda_{y_\kappa^*}^*(t) := \left. \frac{d\nu}{d\pi} \right|_{\mathcal{F}_t^{y_\kappa^*}} = \exp \left(\int_0^t \log \left(\frac{1}{(\kappa \circ y_\kappa^*)'(s)} \right) dY(s) + \int_0^t \left((\kappa \circ y_\kappa^*)'(s) - 1 \right) ds \right). \quad (25)$$

Proof. Remark that under the hypothesis $\dot{y}_\kappa \in \mathcal{D}_2^{++}(\mathcal{F}, \pi_{y_\kappa})$, for almost all sample paths, S_m is infinite after a certain rank thus for any $t \geq 0$,

$$\Lambda_{y_\kappa}(t) = \exp \left(\int_0^t \log \dot{y}_\kappa(s) dN(s) + \int_0^t (1 - \dot{y}_\kappa(s)) \dot{\kappa}(s) ds \right). \quad (26)$$

For $A \in \mathcal{F}_{y_\kappa^*}(t)$, by monotone convergence, we have

$$\begin{aligned} \mathbf{E}_{\pi_{y_\kappa}} [\mathbf{1}_A] &= \lim_{s \rightarrow \infty} \mathbf{E}_{\pi_{y_\kappa}} [\mathbf{1}_A \mathbf{1}_{\{y_\kappa^*(t) \leq s\}}] \\ &= \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}_{\pi_\kappa} [\mathbf{1}_A \mathbf{1}_{\{s < S_n\}} \mathbf{1}_{\{y_\kappa^*(t) \leq s\}} \Lambda_{y_\kappa}] \\ &= \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}_{\pi_\kappa} [\mathbf{1}_A \mathbf{1}_{\{s < S_n\}} \mathbf{1}_{\{y_\kappa^*(t) \leq s\}} \mathbf{E}_{\pi_\kappa} [\Lambda_{y_\kappa}(s) \mid \mathcal{F}_{y_\kappa^*}(t) \wedge s \wedge S_n]] \\ &= \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbf{E}_{\pi_\kappa} [\mathbf{1}_A \mathbf{1}_{\{s < S_n\}} \mathbf{1}_{\{y_\kappa^*(t) \leq s\}} \Lambda_{y_\kappa}(y_\kappa^*(t) \wedge s \wedge S_n)] \end{aligned}$$

according to the stopping time theorem applied to the bounded stopping time $y_\kappa^*(t) \wedge s$ and to the martingale $\Lambda_{y_\kappa}^{S_t}$. Thus, in view of (26), we obtain

$$\mathbf{E}_{\pi_y} [\mathbf{1}_A] = \mathbf{E}_\pi \left[\mathbf{1}_A \Lambda_y \left(y_\kappa^*(t) \right) \right].$$

Then, Theorem 2.6 yields

$$\Lambda_{y_\kappa} \left(y_\kappa^*(t) \right) = \exp \left(\int_0^t \log \dot{y} \left(y_\kappa^*(s) \right) dY(s) + \int_0^t \left(1 - \dot{y} \left(y_\kappa^*(s) \right) \right) \dot{\kappa} \left(y_\kappa^*(s) \right) y_\kappa^*(s) ds \right).$$

On the one hand, we have $y_\kappa' = y \dot{\kappa}$ and on the other hand,

$$y_\kappa \left(y_\kappa^*(s) \right) = s \implies y_\kappa' \left(y_\kappa^*(s) \right) \left(y_\kappa^* \right)'(s) = 1.$$

It follows that

$$\dot{y} \left(y_\kappa^*(s) \right) \dot{\kappa} \left(y_\kappa^*(s) \right) \left(y_\kappa^* \right)'(s) = 1$$

and that

$$\dot{y} \left(y_\kappa^*(s) \right) = \frac{1}{\left(\kappa \circ y_\kappa^* \right)'(s)}.$$

Thus, (25) holds. □

Theorem 2.9 — Quasi-invariance. *Assume that the hypothesis of Lemma 2.8 hold. Then, the distribution of the process $Y = \tau_{y_\kappa}(N)$ under π_{y_κ} is π . This means that for any bounded measurable $f : (\mathfrak{N}, \mathcal{N}_\infty) \rightarrow \mathbf{R}$, for any $t \geq 0$,*

$$\mathbf{E}_{\pi_{y_\kappa}} \left[f \left(Y^t \right) \Lambda_{y_\kappa}^* \left(t \right) \right] = \mathbf{E}_{\pi_{\text{Id}}} [f], \quad (27)$$

where Y^t is the process Y stopped at time t .

Proof. Note that by the stopping time theorem and Theorem 10.27 of [15]

$$\mathbf{E}_{\pi_{y_\kappa}} \left[Y(t) - \kappa \left(y_\kappa^*(t) \right) \mid \mathcal{N}_{y_\kappa^*(s)} \right] = Y(s) - \kappa \left(y_\kappa^*(s) \right),$$

hence the compensator of Y under π_{y_κ} is $\kappa \circ y_\kappa^*$. Thus,

$$\begin{aligned} R(t) &= \int_0^t \frac{1}{\left(\kappa \circ y_\kappa^* \right)'(s)} dY(s) - \int_0^t \frac{1}{\left(\kappa \circ y_\kappa^* \right)'(s)} \left(\kappa \circ y_\kappa^* \right)'(s) ds \\ &= \int_0^t \frac{1}{\left(\kappa \circ y_\kappa^* \right)'(s)} dY(s) - t \end{aligned}$$

is a $(\mathcal{F}^{y_\kappa^*}, \pi_{y_\kappa})$ local martingale. The standard Girsanov theorem [15, Theorem 7.24] states that

$$R(t) - \int_0^t \frac{1}{\Lambda_{y_\kappa}^*(s)} d[R, \Lambda_{y_\kappa}^*](s)$$

is a $(\mathcal{F}^{Y_\kappa^*}, \pi_{Y_\kappa^*})$ local martingale. Note that R and $\Lambda_{Y_\kappa^*}^*$ have the same jump times as Y , hence

$$\begin{aligned} \int_0^t \frac{1}{\Lambda_{Y_\kappa^*}^*(s)} d[R, \Lambda_{Y_\kappa^*}^*](s) &= \sum_{s \leq t, \Delta Y(s) \neq 0} \frac{1}{\Lambda_{Y_\kappa^*}^*(s)} \Delta R(s) \Delta \Lambda_{Y_\kappa^*}^*(s) \\ &= \sum_{s \leq t, \Delta Y(s) \neq 0} \left(1 - \frac{\Lambda_{Y_\kappa^*}^*(s^-)}{\Lambda_{Y_\kappa^*}^*(s)}\right) \Delta R(s) \\ &= \sum_{s \leq t, \Delta Y(s) \neq 0} \left(1 - (\kappa \circ Y_\kappa^*)'(s)\right) \frac{1}{(\kappa \circ Y_\kappa^*)'(s)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} R(t) - \int_0^t \frac{1}{\Lambda_{Y_\kappa^*}^*(s)} d[R, \Lambda_{Y_\kappa^*}^*](s) &= \int_0^t \left[\frac{1}{(\kappa \circ Y_\kappa^*)'(s)} - \left(\frac{1}{(\kappa \circ Y_\kappa^*)'(s)} - 1 \right) \right] dY(s) - t \\ &= Y(t) - t. \end{aligned}$$

This means that Y has $(\mathcal{F}^{Y_\kappa^*}, \pi_{Y_\kappa^*})$ -compensator $(t \mapsto t)$. According to Theorem 2.1, Y is an $\mathcal{F}^{Y_\kappa^*}$ -adapted unit Poisson process under $\pi_{Y_\kappa^*}$. \square

■ **Remark 2** The process $Y_\kappa = (Y(\kappa(t)), t \geq 0)$ is then adapted to the filtration $\mathcal{G} = (\mathcal{N}_{\kappa(t)}^{Y_\kappa^*}, t \geq 0)$ and is a π_κ point process of compensator κ and then we have

$$\mathbf{E}_{\pi_\kappa} \left[f(Y_\kappa^t) \Lambda_{Y_\kappa^*}^*(\kappa(t)) \right] = \mathbf{E}_{\pi_\kappa} [f].$$

This invariance formula is the key to our investigations. In full generality, we could continue in such a general setting with any $\dot{\kappa}$. Actually, changing $\dot{\kappa}$ amounts to change the clock with which time is measured. As long as this clock is deterministic and increasing, this does not change the essence of the results to come so there is no loss in generality but a great gain in simplicity and clarity to focus on the situation where $\kappa = \text{Id}$. For the sake of notations, we suppress the subscript κ hereafter. \blacksquare

The following theorems are often written in terms of \dot{y}^* , this can be translated in terms of \dot{y} as shows the next lemma.

Lemma 2.10 For $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N})$,

$$\begin{aligned} \Lambda_Y &= \exp \left(\int_0^\infty \log(\dot{y}(N, s)) dN(s) + \int_0^\infty (1 - \dot{y}(N, s)) ds \right) \\ &= \Lambda_{Y^*}^* \end{aligned} \tag{28}$$

$$= \exp \left(- \int_0^\infty \log(\dot{y}^*(s)) dY(s) + \int_0^\infty (\dot{y}^*(s) - 1) ds \right) \tag{29}$$

Furthermore, for any $p \geq 1$,

$$\int_0^\infty (\dot{y}^*(s) \log(\dot{y}^*(s)) - \dot{y}^*(s) + 1) ds = \int_0^\infty (\dot{y}(N, s) - 1 - \log(\dot{y}(N, s))) ds \quad (30)$$

$$\int_0^\infty \left| \frac{1}{\dot{y}^*(s)} - 1 \right|^p \dot{y}^*(s) ds = \int_0^\infty |\dot{y}(N, s) - 1|^p ds. \quad (31)$$

Proof. According to Theorem 2.2 point 4 for $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N})$ and to Lemma 2.5, we have

$$\Lambda_y = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_\infty} = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_\infty^{y^*}}.$$

In view of Theorem 8.1 of [15], this means π_y is absolutely continuous with respect to π along the filtration $(\mathcal{N}_t^{y^*}, t \geq 0)$ and (28) follows. Eqn. (29) is a straightforward consequence of (25). Identities (30) and (31), since the integrands are non negative, are obtained through the change of variable $s = y(N, u)$ and the relation

$$\dot{y}^*(y(N, u)) = \frac{1}{\dot{y}(N, u)}.$$

The proof is thus complete. \square

We now prove that well behaved time changes induce locally absolutely continuous probability on \mathfrak{N} . Recall that \mathbf{Y} is defined in Definition 2.8 as the extension to processes of the time change y^* . In the following, we identify the configurations $Y = N \circ y^*$ and $\mathbf{Y}(N)$. The push forward or image measure of π by \mathbf{Y} is denoted by $\mathbf{Y}^\# \pi$.

Theorem 2.11 *Let \dot{y} belong to $\mathcal{D}^{++}(\mathcal{N})$ such that π_y is equivalent to π on \mathcal{N}_∞ . Then $\mathbf{Y}^\# \pi$ is equivalent to π on \mathcal{N}_∞ .*

Proof. Since π_y is equivalent to π on \mathcal{N}_∞ , $\mathbf{Y}^\# \pi_y$ is equivalent to $\mathbf{Y}^\# \pi$.

Furthermore, we have

$$\Lambda_{y^*(t)}^* = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_{y^*(t)}},$$

hence,

$$\lim_{t \rightarrow \infty} \Lambda_{y^*(t)}^* = \frac{d\pi_y}{d\pi} \Big|_{\mathcal{N}_\infty} = \Lambda_y.$$

According to Lemma 2.3, the martingale

$$\left(\Lambda_{y^*(t)}^*, t \geq 0 \right)$$

is then uniformly integrable and we can let t go to infinity in (27) to obtain

$$\mathbf{Y}^\# \pi_y = \pi.$$

As a consequence,

$$\pi = \mathbf{Y}^\# \pi_y \sim \mathbf{Y}^\# \pi.$$

The proof is thus complete. \square

3 Invertibility

We now define the notion of left and right invertibility we will analyze in the following. We introduce a new notation for maps from \mathfrak{N} into itself as in full generality, such a map is not necessarily induced by a time change γ . We write in bold the transformation when it is associated to time change and in blackboard bold for an abstract transformation of \mathfrak{N} . Note that we must be careful when we compose random variables: If R and \tilde{R} are random variable from Ω to a space E which are equal \mathbf{P} -a.s., we must ensure that for S and \tilde{S} which are \mathbf{P} -almost surely equal random variables from Ω to Ω , we still have

$$\mathbf{P}(R \circ S \neq \tilde{R} \circ \tilde{S}) = 0. \quad (32)$$

But,

$$\mathbf{P}(R \circ S \neq \tilde{R} \circ \tilde{S}) = \mathbf{P}(R \circ S \neq \tilde{R} \circ S) = \mathbf{P}_S(R \neq \tilde{R}).$$

So a sufficient condition for (32) to hold is that $\mathbf{P}_S \ll \mathbf{P}$.

Definition 3.1 Let $(\mathfrak{N}, \mu, \mathcal{F}_\infty)$ be a probability space and \mathbb{Y} a map from \mathfrak{N} to \mathfrak{N} .

The map \mathbb{Y} is left invertible if and only if $\mathbb{Y}^\# \mu \ll \mu$ and there exists $\mathbb{Z} : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $\mathbb{Z} \circ \mathbb{Y} = \text{Id}_{\mathfrak{N}}$, μ -a.s.

The map \mathbb{Y} is right invertible if and only if there exists $\mathbb{Z} : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $\mathbb{Z}^\# \mu \ll \mu$ and $\mathbb{Y} \circ \mathbb{Z} = \text{Id}_{\mathfrak{N}}$, μ -a.s.

The map \mathbb{Y} is invertible if it is both left and right invertible.

Lemma 3.1 *If there exists \mathbb{Z} such that $\mathbb{Z} \circ \mathbb{Y} = \text{Id}_{\mathfrak{N}}$, μ -a.s. then $\mathbb{Y} \circ \mathbb{Z} = \text{Id}_{\mathfrak{N}}$, $\mathbb{Y}^\# \mu$ -a.s.*

If additionally, $\mathbb{Y}^\# \mu$ is equivalent to μ and $\mathbb{Z}^\# \mu \ll \mu$, then \mathbb{Y} is invertible and $\mathbb{Z}^\# \mu$ is equivalent to μ .

Proof. We have

$$\begin{aligned} \mathbb{Y}^\# \mu(\mathbb{Y} \circ \mathbb{Z} = \text{Id}_{\mathfrak{N}}) &= \mu(\mathbb{Y} \circ \mathbb{Z} \circ \mathbb{Y} = \mathbb{Y}) \\ &= \mu(\mathbb{Y} = \mathbb{Y}) \\ &= 1. \end{aligned}$$

The first assertion follows. If the two measures $\mathbb{Y}^\# \mu$ and μ are equivalent, then $\mathbb{Y} \circ \mathbb{Z} = \text{Id}_{\mathfrak{N}}$ μ -almost-surely thus \mathbb{Y} is invertible.

Let A such that $\mathbb{Z}^\# \mu(A) = 0$. This means

$$\mathbf{E}_\mu[\mathbf{1}_A \circ \mathbb{Z}] = 0.$$

Since $\mathbb{Y}^\# \mu$ is equivalent to μ , we get

$$0 = \mathbf{E}_\mu[\mathbf{1}_A \circ \mathbb{Z} \circ \mathbb{Y}] = \mathbf{E}_\mu[\mathbf{1}_A]$$

hence $\mu \ll \mathbb{Z}^\# \mu$ and the equivalence follows. \square

We now state a technical lemma which indicates how the time changes of two inverse maps are related to each other.

Lemma 3.2 Let $\dot{y} \in \mathcal{D}^{++}(\mathcal{F})$ such that $\mathbf{Y}^\# \pi \ll \pi$. For z a π -a.s. positive process such that $z^*(Y, t)$ is an \mathcal{F}^{y^*} -stopping time, we have

$$(\mathbf{Z} \circ \mathbf{Y})(t) = N\left(y^*(N, z^*(Y, t))\right). \quad (33)$$

Then \mathbf{Z} is the left inverse of \mathbf{Y} if and only if

$$z^*(Y, t) = y(N, t), \quad \forall t, \pi\text{-a.s.} \quad (34)$$

or equivalently

$$z(Y, t) = y^*(N, t), \quad \forall t, \pi\text{-a.s.} \quad (35)$$

Proof. The first part comes from the identities:

$$\begin{aligned} (\mathbf{Z} \circ \mathbf{Y})(N)(t) &= Y\left(z^*(Y, t)\right) \\ &= N\left(y^*(N, z^*(Y, t))\right). \end{aligned}$$

Thus,

$$\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{N}} \iff y^*(N, z^*(Y, t)) = t, \quad \forall t, \pi\text{-a.s.}$$

According to (17), this entails (34). \square

We have already seen in Lemma 2.5 that \mathcal{N}^{y^*} is larger than \mathcal{Y} . Actually, there is a stronger result:

Theorem 3.3 Let $\dot{y} \in \mathcal{D}^{++}(\mathcal{N}, \pi)$ such that $\mathbf{Y}^\# \pi \ll \pi$. Then, $\mathcal{N}^{y^*} = \mathcal{Y}$ if and only if \mathbf{Y} admits a left inverse.

Proof. If \mathbf{Y} is left-invertible then there exists z such that (35) holds :

$$z(Y, t) = y^*(N, t), \quad \forall t, \pi\text{-a.s.}$$

This means that $y^*(N, s)$ is $\mathcal{Y}_s \subset \mathcal{Y}_t$ measurable, thus $\mathcal{Y}_t \vee \sigma(y^*(s), s \leq t) = \mathcal{Y}_t$.

Conversely, if $\mathcal{N}_t^{y^*} = \mathcal{Y}_t$ then as $y^*(N, t)$ is $\mathcal{N}_t^{y^*}$ measurable, it is also \mathcal{Y}_t -measurable. Hence, for any $t \geq 0$, there exists a random variable $\tilde{z}(Y^t, t)$ such that

$$\tilde{z}(Y^t, t) = y^*(N, t), \quad \pi\text{-a.s.}$$

We can then find a full probability set A such that

$$\tilde{z}(Y^t(\omega), t) = y^*(\omega, t), \quad \forall t \in \mathbf{Q}, \forall \omega \in A. \quad (36)$$

Let

$$z(Y^t(\omega), t) = \begin{cases} y^*(\omega, t) & \text{if } t \in \mathbf{Q} \\ \lim_{r_n \rightarrow t, r_n \in \mathbf{Q}} y^*(\omega, r_n) & \text{if } t \notin \mathbf{Q}. \end{cases}$$

By the sample path continuity of y^* , (36) holds for any $t \in \mathbf{R}^+$ with probability 1. Hence, Lemma 3.2 implies that \mathbf{Y} admits a left inverse. \square

We can now state the link between right invertibility of time change on \mathfrak{N} and existence of g-Hawkes processes.

Theorem 3.4 *Let $\dot{y} \in \mathcal{D}^{++}(\mathcal{N})$. Then, \mathbf{Y} is right invertible with a right inverse of the form $Z = N(z^*)$ where for any $t > 0$, $z^*(N, t)$ is an \mathcal{N} -stopping time if and only if there exists a process Z , whose law is absolutely continuous with respect to π and has compensator $y(Z, \cdot)$, i.e. Z is a solution of the equation*

$$Z(t) = N\left(y(Z, t)\right) \quad (37)$$

with the additional constraint that $y(Z, t)$ is an \mathcal{N} -stopping time.

Proof. If \mathbf{Y} is right invertible then $\mathbf{Z}^\# \pi \ll \pi$, hence it is meaningful to define the process $(y(Z(N), \cdot), t \geq 0)$. If $Z = N(z^*)$ is the right inverse of \mathbf{Y} , then, according to (35), we have

$$z^*(N, t) = y(Z(N), t), \pi - \text{a.s.}$$

so that, if we set

$$Z(t) := N\left(z^*(N, t)\right),$$

we get

$$Z(t) = N\left(y(Z(N), t)\right).$$

This means that Z satisfies (37). Since $z^*(N, t)$ is an \mathcal{N} -stopping time, so does $y(Z, t)$ and the additional constraint is altogether satisfied.

Conversely, if Z satisfies (37) and $y(Z, t)$ is an \mathcal{N} -stopping time for any $t \geq 0$, we set

$$z^*(N, t) = y(Z, t)$$

so that $z^*(N, t)$ is a stopping time for any $t \geq 0$. Furthermore, according to (34) we have $\mathbf{Y} \circ \mathbf{Z} = \text{Id}_{\mathfrak{N}}$, which, with the hypothesis $\mathbf{Z}^\# \pi \ll \pi$, means that \mathbf{Y} is right invertible. \square

Corollary 3.5 *If Z satisfies (37) and $y(Z, t)$ is an \mathcal{N} stopping time for any $t \geq 0$, then*

$$\mathcal{Z}_t = \mathcal{N}_{y(Z, t)}$$

where \mathcal{Z} is the σ -field generated by the sample path of Z . Furthermore, $\mathbf{Z}^\# \pi = \pi_y$.

Proof. Actually, Theorem 3.4 implies that \mathbf{Z} is left invertible and the conclusion follows by Theorem 3.3. Consequently, according to (37), Z has $y(Z, \cdot)$ as a (π, \mathcal{Z}) -compensator. This means that the law of Z has the same compensator with respect to its minimal filtration as N under π_y . Theorem 2.1 then entails that $\mathbf{Z}^\# \pi = \pi_y$. \square

Corollary 3.6 *Let $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$. Assume that \mathbf{Y} is right invertible and let Z be the g-Hawkes process defined in Theorem 3.4. Then, Z admits the martingale representation property: for any $F \in L^2(\mathfrak{N} \rightarrow \mathbf{R}, \mathbf{Z}^\# \pi)$, there exists v a \mathcal{Z} -predictable process such that,*

π -a.s., we have

$$F \circ \mathbf{Z} = \mathbf{E}_{\mathbf{Z}^\# \pi} [F] + \int_0^\infty v(Z, s) (dZ(s) - \dot{y}(Z, s) ds)$$

where

$$v(N, s) = u\left(Y, y(N, s)\right).$$

Note that v satisfies

$$\mathbf{E}_\pi \left[\int_0^\infty v(Z, s)^2 \dot{y}(Z, s) ds \right] < \infty.$$

Proof. Since \mathbf{Z} is the right inverse of \mathbf{Y} , \mathbf{Y} is the left inverse of \mathbf{Z} . According to Theorem 3.3, $\mathcal{Z} = \mathcal{N}^{\mathbf{Z}^*}$. By the very definition of right invertibility, $\mathbf{Z}^\# \pi \ll \pi$ hence for F a random variable on (\mathfrak{N}, π) , the random variable $F_Z := F \circ \mathbf{Z}$ is well defined. When F_Z is π -square integrable, we know from the martingale representation property for the Poisson process [16] that there exists a square integrable, predictable process u such that

$$F_Z = \mathbf{E}_\pi [F_Z] + \int_0^\infty u(s) (dN(s) - ds).$$

The relation (35) means that

$$T_k(N) = y(Z, T_k(Z)).$$

We obtain

$$\begin{aligned} \int_0^\infty u(s) dN(s) &= \sum_{k \geq 1} u(N, T_k(N)) \\ &= \sum_{k \geq 1} u\left(Y(Z), y(Z, T_k(Z))\right) \\ &= \int_0^\infty \left(u(Y, \cdot) \circ \mathbf{Z}\right)(s) dZ(s). \end{aligned}$$

We know that Z has $(\pi, \mathcal{N}^{\mathbf{Z}^*})$ compensator $Z^p(t) = y(Z, t)$ hence this process is also the compensator of Z for the filtration \mathcal{Z} . The change of variable $s = y(Z, r)$ yields

$$\begin{aligned} \int_0^\infty u(N, s) ds &= \int_0^\infty u\left(N, y(Z, r)\right) \dot{y}(Z, r) dr \\ &= \int_0^\infty v(Z, s) dZ^p(r). \end{aligned}$$

The proof is thus complete. □

Definition 3.2 For μ and ν two probability measures on \mathfrak{N} , the relative entropy of ν with respect to μ is given by

$$H(\nu | \mu) = \begin{cases} \mathbf{E}_\nu \left[\log \left(\frac{d\nu}{d\mu} \Big|_{\mathcal{N}_\infty} \right) \right] & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

The next theorem is a crucial step towards the final proof. Roughly speaking, given π_z , we aim to find a time change y^* such that $\mathbf{Y}^\# \pi = \pi_z$. The necessary condition is that y^* solves (40) which implies $\Lambda_y = 1/\Lambda_z \circ \mathbf{Y}$. Following the proof of [18, Proposition 2.1], we have:

Theorem 3.7 *Let $z \in \mathcal{P}^{++}(\mathcal{N})$. Assume π_z is equivalent to π on \mathcal{N}_∞ and that there exists $\dot{y} \in \mathcal{P}_2^{++}(\mathcal{N}, \pi)$ such that $\mathbf{Z} \circ \mathbf{Y} = I_{\mathfrak{Y}} \pi$ -a.s. Then the following assertions are equivalent:*

(i) *We have*

$$\mathbf{E}_\pi [\Lambda_y] = 1 \text{ and } \pi(\Lambda_y > 0) = 1. \quad (38)$$

(ii) *$\mathbf{Y}^\# \pi$ is equivalent to π on \mathcal{N}_∞ .*

(iii) *\mathbf{Z} is left invertible with inverse \mathbf{Y} .*

(iv) *$\mathbf{Y}^\# \pi = \pi_z$ on \mathcal{N}_∞ and we have the following identity*

$$\log(\Lambda_z \circ \mathbf{Y}) = \int_0^\infty \log(\dot{y}^*(N, s)) dY(s) + \int_0^\infty (1 - \dot{y}^*(N, s)) ds, \quad \pi - \text{a.s.} \quad (39)$$

Proof. (i) \implies (ii). According to Lemma 2.3, π_y is equivalent to π on \mathcal{N}_∞ . Apply Theorem 2.11 and point (ii) follows.

(ii) \implies (iii). According to Lemma 3.1, $\mathbf{Y} \circ \mathbf{Z} = \text{Id}_{\mathfrak{Y}}$, $\mathbf{Y}^\# \pi$ a.s. In view of (ii), this identity holds π almost surely. Moreover, still according to (ii),

$$\mathbf{Z}^\# \pi \sim \mathbf{Z}^\# (\mathbf{Y}^\# \pi) = \pi$$

and (iii) follows from Lemma 3.1.

(iii) \implies (iv). It follows from (iii) that $\pi_z = (\mathbf{Y} \circ \mathbf{Z})^\# \pi_z = \mathbf{Y}^\# (\mathbf{Z}^\# \pi_z)$. Recall that the quasi-invariance theorem says that $\mathbf{Z}^\# \pi_z = \pi$ hence $\mathbf{Y}^\# \pi = \pi_z$.

According to (23) and (24),

$$\Lambda_{z^*} \circ \mathbf{Y} = \exp \left(- \int_0^\infty \log(\dot{z}^*(Y, y^*(N, s))) dY(s) + \int_0^\infty (\dot{z}^*(Y, y^*(N, s)) - 1) \dot{y}^*(N, s) ds \right).$$

Since $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{Y}}$, according to (34) we have

$$z^*(Y, y^*(N, t)) = t.$$

By differentiation, we obtain

$$\dot{z}^*(Y, y^*(N, t)) \times \dot{y}^*(N, t) = 1, \quad \pi - \text{a.s.} \quad (40)$$

We obtain

$$\log(\Lambda_z \circ \mathbf{Y}) = \int_0^\infty \log(\dot{y}^*(N, s)) dY(s) + \int_0^\infty \left(\frac{1}{\dot{y}^*(N, s)} - 1 \right) \dot{y}^*(N, s) ds$$

and (39) follows

(iv) \implies (i). Recall that

$$\Lambda_y = \exp \left(- \int_0^\infty \log(\dot{y}^*(N, s)) dY(s) - \int_0^\infty (1 - \dot{y}^*(N, s)) ds \right).$$

Equation (39) amounts to

$$\Lambda_y = \frac{1}{\Lambda_{z^*}} \circ \mathbf{Y} = \frac{1}{\Lambda_z} \circ \mathbf{Y}.$$

Since $\mathbf{Y}^\# \pi = \pi_z$, we obtain

$$\begin{aligned} \mathbf{E}_\pi [\Lambda_y] &= \mathbf{E}_{\pi_z} \left[\frac{1}{\Lambda_z} \right] \\ &= \mathbf{E}_\pi \left[\frac{1}{\Lambda_z} \Lambda_z \right] = 1. \end{aligned}$$

Moreover, still from (39), we deduce that

$$\begin{aligned} \pi(\Lambda_y = 0) &= \pi \left(\frac{1}{\Lambda_z} \circ \mathbf{Y} = 0 \right) \\ &= \mathbf{Y}^\# \pi(\Lambda_z = +\infty) \\ &= \pi_z(\Lambda_z = +\infty). \end{aligned}$$

According to the hypothesis, π_z is equivalent to π and Lemma 2.3 entails that $\mathbf{E}_\pi[\Lambda_z]$ is finite hence $\pi(\Lambda_z = +\infty) = 0$ and then $\pi_z(\Lambda_z = +\infty) = 0$. It follows that

$$\pi(\Lambda_y = 0) = 0,$$

so that (i) holds. □

Lemma 3.8 *Let $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$ such that $\mathbf{E}_\pi[\Lambda_y] = 1$ and*

$$\mathbf{E}_\pi \left[\int_0^\infty m(\dot{y}^*(N, s) - 1) ds \right] < \infty.$$

Then,

$$\mathbf{E}_\pi \left[-\log \Lambda_{y^*} \right] \leq \mathbf{E}_\pi \left[\int_0^\infty m(\dot{y}(N, s) - 1) ds. \right] \quad (41)$$

Proof. We have already seen that

$$\dot{y}^*(t) = \frac{1}{\dot{y}(y^*(t))} = \tau_{y^*} \left(\frac{1}{\dot{y}} \right) (t).$$

By hypothesis, \dot{y} is \mathcal{N} -predictable, thus, according to [15, Theorem 10.17(c)], \dot{y}^* is predictable with respect to the filtration \mathcal{N}^{y^*} .

From (29), we have

$$\begin{aligned}\mathbf{E}\left[-\log\Lambda_{y^*}^*(t)\right] &= \mathbf{E}\left[\int_0^t \log(\dot{y}^*(s)) dY(s) + \int_0^t (1 - \dot{y}^*(s)) ds\right] \\ &= \mathbf{E}\left[\int_0^t \dot{y}^*(s) \log(\dot{y}^*(s)) + 1 - \dot{y}^*(s) ds\right] \\ &\leq \mathbf{E}\left[\int_0^\infty m(\dot{y}^*(s) - 1) ds\right],\end{aligned}$$

since Y has \mathcal{N}^{y^*} compensator y^* and $\log \dot{y}^*$ is \mathcal{N}^{y^*} predictable. It remains to prove that we can pass to the limit in the left-hand-side. Consider the non-negative, convex function $\psi(x) = x - \log x$. From Fatou's Lemma, we have

$$\begin{aligned}\mathbf{E}\left[\psi(\Lambda_{y^*}^*)\right] &\leq \liminf_{t \rightarrow \infty} \mathbf{E}\left[\psi(\Lambda_{y^*}^*(t))\right] \\ &\leq 1 + \mathbf{E}\left[\int_0^\infty m(\dot{y}^*(s) - 1) ds\right].\end{aligned}$$

This means that the non-negative submartingale $(\psi(\Lambda_{y^*}^*(t)), t \geq 0)$ is uniformly integrable. From (28), we know that $\Lambda_y = \Lambda_{y^*}^*$ hence in view of the first hypothesis, the non-negative martingale $(\Lambda_{y^*}^*(t), t \geq 0)$ is uniformly integrable,

$$-\log\Lambda_{y^*}^*(t) \xrightarrow[t \rightarrow \infty]{L^1} -\log\Lambda_{y^*}^*.$$

This means that

$$1 + \mathbf{E}\left[-\log\Lambda_{y^*}^*\right] \leq 1 + \mathbf{E}\left[\int_0^\infty m(\dot{y}^*(s) - 1) ds\right].$$

The proof is thus complete. \square

We arrive now at the main result of this section, the entropic criterion of (left) invertibility. Before going into the details of the proof, we explain its main idea. At the very beginning, we are given y an increasing predictable process whose inverse can be used as a time change. From y , one can construct Λ_y and $\mathbf{Y}^\# \pi$. Note carefully that Λ_y is not the density of $\mathbf{Y}^\# \pi$ with respect to π . Actually, under the hypothesis we made on y , $\mathbf{Y}^\# \pi$ is absolutely continuous with respect to π and there exists z such that

$$\frac{d\mathbf{Y}^\# \pi}{d\pi} = \Lambda_z.$$

If we have

$$\log\Lambda_z \circ \mathbf{Y} = -\log\Lambda_y, \tag{42}$$

then, from the representations of these quantities (see Theorem 2.7), we see that

$$z^*(Y, y^*(N, t)) = 1,$$

which according to Lemma 3.2 means that $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{Y}}$. Before going further, recall that at the terminal time $\Lambda_y = \Lambda_{y^*}^*$ but at time t , $\Lambda_y(t)$ is \mathcal{N}_t measurable whereas $\Lambda_{y^*}^*(t)$ is $\mathcal{N}_{y^*(N, t)}$

adapted. For the sake of clarity, we write all the conditions in terms of Λ_y even if in the following proofs we need to use Λ_{y^*} .

Now, on the one hand, quasi-invariance and Fatou lemma induce that

$$\log \Lambda_z \circ \mathbf{Y} \leq -\log \mathbf{E}_\pi [\Lambda_y | \mathcal{Y}_\infty]. \quad (43)$$

On the other hand, classical computations and Jensen inequality show that

$$\begin{aligned} H(\mathbf{Y}^\# \pi | \pi) &= \mathbf{E}_\pi [\log \Lambda_z \circ \mathbf{Y}] \leq -\mathbf{E}_\pi [\log \mathbf{E}_\pi [\Lambda_y | \mathcal{Y}_\infty]] \\ &\leq -\mathbf{E}_\pi [\log \Lambda_y] = \mathbf{E}_\pi \left[\int_0^\infty \mathfrak{m}(\dot{y}^*(s) - 1) \, ds \right]. \end{aligned} \quad (44)$$

If the entropic criterion is satisfied, then all the inequalities of (44) are indeed transformed into equalities. The equality condition in the conditional Jensen inequality implies that

$$\mathbf{E}_\pi [\Lambda_y | \mathcal{Y}_\infty] = \Lambda_y, \quad (45)$$

which in turn entails that (43) becomes

$$\log \Lambda_z \circ \mathbf{Y} \leq -\log \Lambda_y.$$

In view of the first inequality of (44), we obtain the identity (42) and the invertibility follows.

Remark that (45) says that Λ_y , which is a priori $\mathcal{N}_\infty = \mathcal{N}_\infty^{y^*}$ measurable, is actually \mathcal{Y}_∞ measurable which in view of Theorem 3.3, is a necessary condition for \mathbf{Y} to be left-invertible.

Theorem 3.9 *Let $\dot{y} \in \mathcal{P}_2^{++}(\mathcal{N}, \pi_y)$ such that $\mathbf{E}_\pi [\Lambda_y] = 1$ and*

$$\mathbf{E}_\pi \left[\int_0^\infty \mathfrak{m}(\dot{y}^*(s) - 1) \, ds \right] < \infty.$$

If $\mathbf{Y}^\# \pi \ll \pi$, we have

$$H(\mathbf{Y}^\# \pi | \pi) \leq \mathbf{E} \left[\int_0^\infty \mathfrak{m}(\dot{y}^*(s) - 1) \, ds \right]. \quad (46)$$

Moreover, the map \mathbf{Y} is left invertible if and only if

$$H(\mathbf{Y}^\# \pi | \pi) = \mathbf{E}_\pi \left[\int_0^\infty \mathfrak{m}(\dot{y}^*(s) - 1) \, ds \right]. \quad (47)$$

■ **Remark 3** Following [2], the function $x \mapsto \mathfrak{m}(x - 1)$ plays in the Poisson settings the exact same role as the function $x \mapsto x^2/2$ in the Gaussian world. Since the entropic criterion in the Wiener reads as

$$H(U^\# \mu | \mu) = \frac{1}{2} \mathbf{E}_\mu \left[\int_0^1 \dot{u}(s)^2 \, ds \right]$$

where U is defined in (2), we see that the Gaussian-Poisson analogy alluded to in [2] goes even further. ■

Proof. Since $\mathbf{Y}^\# \pi \ll \pi$, Theorem 2.2 implies that there exists $\dot{z} \in \mathcal{D}_2^+(\mathcal{N}, \mathbf{Y}^\# \pi)$ such that

$$\left. \frac{d\mathbf{Y}^\# \pi}{d\pi} \right|_{\mathcal{N}_\infty} = \Lambda_z.$$

The quasi-invariance Theorem 2.9 then says that for each $t > 0$, for $f : \mathfrak{N} \rightarrow \mathbf{R}^+$ bounded and continuous,

$$\begin{aligned} \mathbf{E}_\pi [f \circ \mathbf{Y}^t] &= \mathbf{E}_\pi [f(Y^t)] \\ &= \mathbf{E}_\pi [f \Lambda_z(t)] \\ &= \mathbf{E}_\pi [f \circ \mathbf{Y}^t \Lambda_z \circ \mathbf{Y}^t(t) \Lambda_{y^*}^*(t)]. \end{aligned}$$

By Fatou's Lemma, we obtain

$$\mathbf{E}_\pi [f \circ \mathbf{Y} \Lambda_z \circ \mathbf{Y} \Lambda_{y^*}^*] \leq \mathbf{E}_\pi [f \circ \mathbf{Y}]$$

or equivalently

$$\mathbf{E}_\pi [f \circ \mathbf{Y} \Lambda_z \circ \mathbf{Y} \mathbf{E} [\Lambda_{y^*}^* | \mathcal{Y}_\infty]] \leq \mathbf{E}_\pi [f \circ \mathbf{Y}]$$

Hence, π -a.s. we have

$$\Lambda_z \circ \mathbf{Y} \times \mathbf{E}_\pi [\Lambda_{y^*}^* | \mathcal{Y}_\infty] \leq 1. \quad (48)$$

It follows that

$$\begin{aligned} 0 \leq H(\mathbf{Y}^\# \pi | \pi) &= \mathbf{E}_\pi [\log \Lambda_z \circ \mathbf{Y}] \\ &\leq -\mathbf{E}_\pi [\log \mathbf{E}_\pi [\Lambda_{y^*}^* | \mathcal{Y}_\infty]]. \end{aligned} \quad (49)$$

Since $-\log$ is convex, the Jensen inequality stands that

$$\begin{aligned} H(\mathbf{Y}^\# \pi | \pi) &\leq -\mathbf{E}_\pi [\log \Lambda_{y^*}^*] \\ &= \mathbf{E}_\pi \left[\int_0^\infty m(y^*(s) - 1) ds \right], \end{aligned} \quad (50)$$

according to Lemma 3.8. Then the first part holds.

Assume now that (47) holds. Then (49) and (50) are indeed equalities. On the one hand, this means that we have equality in the Jensen inequality used to derive (50). Since $-\log$ is strictly convex, it follows that [21, Cor. 2.1]:

$$\mathbf{E}_\pi [\Lambda_{y^*}^* | \mathcal{Y}_\infty] = \Lambda_{y^*}^*, \quad \pi - \text{a.s.}$$

On the other hand, this also implies that (49) is an equality and as a consequence of (48), we obtain

$$\log \Lambda_z^* \circ \mathbf{Y} = -\log \Lambda_{y^*}^*. \quad (51)$$

According to (23) and (24),

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = - \int_0^\infty \log \left(\dot{z}^*(Y, y^*(N, s)) \right) dY(s) + \int_0^\infty \left(\dot{z}^*(Y, y^*(N, s)) - 1 \right) \dot{y}^*(N, s) ds$$

and

$$- \log \Lambda_{y^*}^* = \int_0^\infty \log \left(\dot{y}^*(N, s) \right) dY(s) - \int_0^\infty \left(\dot{y}^*(N, s) - 1 \right) ds.$$

Hence, according to (47), we can take the conditional expectation with respect to \mathcal{Y}_t in both equalities to obtain:

$$\begin{aligned} - \int_0^t \log \left(\dot{z}^*(Y, y^*(N, s)) \right) dY(s) + \int_0^t \left(\dot{z}^*(Y, y^*(N, s)) - 1 \right) \dot{y}^*(N, s) ds \\ = \int_0^t \log \left(\dot{y}^*(N, s) \right) dY(s) - \int_0^t \left(\dot{y}^*(N, s) - 1 \right) ds. \end{aligned}$$

Equating the jumps yields

$$\dot{z}^*(Y, y^*(N, s)) \times \dot{y}^*(N, s) = 1,$$

then, by integration, we get

$$z^*(Y, y^*(N, t)) = t.$$

Furthermore,

$$(z^*(Y, t) \leq s) = (y(N, t) \leq s) = (y^*(N, s) \geq t) \in \mathcal{N}_s^{y^*},$$

hence $z^*(Y, t)$ is an \mathcal{N}^{y^*} stopping time. According to (34), this means that $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{Y}}$ and then \mathbf{Y} is left invertible.

Conversely, if \mathbf{Y} is left invertible. According to the Definition 3.1, $\mathbf{Y}^\# \pi$ is absolutely continuous with respect to π . Let us denote by \mathbb{Z} the map such that

$$\mathbb{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{Y}}, \pi - \text{a.s.}$$

Define the process z by

$$z(N, t) = y^*(\mathbb{Z}(N), t), \forall t \geq 0.$$

Since $\mathbf{Y}^\# \pi \ll \pi$,

$$\mathbf{Y}^\# \pi (z(N, \cdot) = y^*(\mathbb{Z}(N), \cdot)) = 1$$

and

$$\pi (z(Y, \cdot) = y^*(N, \cdot)) = \mathbf{Y}^\# \pi (z(N, \cdot) = y^*(\mathbb{Z}(N), \cdot)).$$

This means that

$$z(Y, t) = y^*(N, t), \forall t \geq 0, \pi\text{-a.s.}$$

or otherwise stated, $\mathbf{Z} \circ \mathbf{Y} = \text{Id}_{\mathfrak{Y}}$, π -a.s. Moreover, by differentiation, we get

$$\dot{z}^*(Y, y^*(N, t)) = \frac{1}{\dot{y}^*(N, t)}, \forall t \geq 0, \pi - \text{a.s.} \quad (52)$$

Since $\dot{y}(N, \cdot)$ belongs to $\mathcal{D}_2^{++}(\mathcal{N}, \pi)$, so does $\dot{z}^*(Y, \cdot)$, hence we can write

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = - \int_0^\infty \log \left(\dot{z}^*(Y, y^*(N, s)) \right) dY(s) + \int_0^\infty \left(\dot{z}^*(Y, y^*(N, s)) - 1 \right) \dot{y}^*(N, s) ds,$$

and (52) entails that

$$\log \Lambda_{z^*}^* \circ \mathbf{Y} = - \log \Lambda_{y^*}^*. \quad (53)$$

Let

$$R = \frac{d\mathbf{Y}^\# \pi}{d\pi}.$$

For any $f : \mathfrak{N} \rightarrow \mathbf{R}$ continuous and bounded, for any $t > 0$, we have

$$\begin{aligned} \mathbf{E}_\pi [fR] &= \mathbf{E}_\pi [f \circ \mathbf{Y}] \\ &= \mathbf{E}_\pi \left[(f \Lambda_{z^*}^*) \circ \mathbf{Y} \Lambda_{y^*}^* \right] \\ &= \mathbf{E}_\pi [f \Lambda_{z^*}^*] \end{aligned}$$

according to (53) and to the quasi-invariance Theorem. It follows that $R = \Lambda_{z^*}^*$, π -a.s. Plug this identity into (53) to obtain

$$\begin{aligned} H(\mathbf{Y}^* \pi | \pi) &= \mathbf{E}_\pi [\log R \circ \mathbf{Y}] \\ &= \mathbf{E}_\pi [\log \Lambda_{z^*}^* \circ \mathbf{Y}] \\ &= \mathbf{E}_\pi [-\log \Lambda_{y^*}^*] \\ &= \mathbf{E}_\pi \left[\int_0^\infty m(\dot{y}^*(s) - 1) ds \right]. \end{aligned}$$

The entropic criterion is thus satisfied. \square

4 Variational representation of the entropy

We now give an interesting application of the previous considerations where the entropic criterion is the key to the approximation procedure needed in the proof of this representation of the entropy.

Let

$$\begin{aligned} \mathcal{D}_m^{++} &= \{y, \dot{y} \in \mathcal{D}^{++}(\mathcal{N}) \text{ and } (\dot{y} - 1) \in L^1(\mathfrak{N}; \mathbf{L}_m, \pi)\} \\ \mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi) &= \mathcal{D}_m^{++}(\mathcal{N}, \pi) \cap \{y, \dot{y} \text{ piecewise constant}\} \\ \mathfrak{M}_m(\mathfrak{N}) &= \{\mu, \exists y \in \mathcal{D}_m^{++} \text{ such that } \mu = \mathbf{Y}^\# \pi\}. \end{aligned}$$

The first step of the proof consists in proving the existence of a g-Hawkes process for a piecewise constant time change (see [12, 23] for the Brownian analog).

Lemma 4.1 Let $\dot{y} \in \mathcal{P}_{\infty}^{++}(\mathcal{N}, \pi)$ be piecewise constant: If we denote by T the time after which $\dot{y}(N, s) = 1$, consider a partition of $[0, T]$, $0 = t_0 < t_1 < \dots < t_k = T < t_{k+1} = +\infty$ and assume that there exist $\alpha_0 \in \mathbf{R}^+$ and some random variables $(\alpha_j, j = 1, \dots, k)$ such that for some $\epsilon > 0$,

$$\epsilon \leq \alpha_j(N) \leq 1/\epsilon, \quad \forall j = 0, \dots, k$$

and

$$\dot{y}(N, s) = \alpha_0 \mathbf{1}_{(0, t_1]}(s) + \sum_{j=1}^{k-1} \alpha_j (N^{t_j}) \mathbf{1}_{(t_j, t_{j+1}]}(s) + \alpha_k (N^T) \mathbf{1}_{[T, \infty)}(s).$$

Then, \mathbf{Y} is invertible.

Proof. We first prove that \mathbf{Y} is right invertible. The g-Hawkes process Z is constructed inductively. On $[0, t_1]$, we set

$$Z(t) = N(\alpha_0 t).$$

Then,

$$y(Z, t) = \alpha_0 t$$

and we do have $Z(t) = N(y(Z, t))$. For any $t \leq t_1$, $y(N, t)$ is deterministic hence it is an \mathcal{N} -stopping time.

Assume that Z is constructed on $[0, t_m]$ with $m < k$ and $y(Z, t)$ is an \mathcal{N} -stopping time for $t \leq t_m$. For $t \in [t_m, t_{m+1}]$, we have

$$y(Z, t) = y(Z, t_m) + \alpha_m (Z^{t_m})(t - t_m). \quad (54)$$

By the induction hypothesis, $y(Z, t_m)$ is an \mathcal{N} -stopping time hence the σ -field $\mathcal{N}_{y(Z, t_m)}$ is well defined and $y(Z, t_m)$ belongs to this σ -field. Furthermore, for $t \leq t_m$,

$$Z(t) = N(y(Z, t)) \in \mathcal{N}_{y(Z, t_m)}.$$

It follows that for $t \in [t_m, t_{m+1}]$,

$$y(Z, t) \in \mathcal{N}_{y(Z, t_m)}.$$

Since $\alpha_m > 0$, for any $s \geq 0$, according to (54),

$$\begin{aligned} \left(y(Z, t) \leq s \right) &= \left(\alpha_m (Z^{t_m}) \leq \frac{s - y(t_m)}{t - t_m} \right) \\ &= \left(\alpha_m (Z^{t_m}) \leq \frac{s - y(t_m)}{t - t_m} \right) \cap (y(Z, t_m) \leq s) \end{aligned}$$

which belongs to \mathcal{N}_s by the definition of $\mathcal{N}_{y(Z, t_m)}$. Hence $y(Z, t)$ is an \mathcal{N} -stopping time for $t \leq t_{m+1}$. Moreover, (54) guarantees that there is no ambiguity to define Z on $[t_m, t_{m+1}]$ by $Z(t) = N(y(Z, t))$. According to Theorem 3.4, \mathbf{Y} is right invertible.

Remark that Z has (\mathcal{N}, π) compensator $y(Z, \cdot)$. In view of Theorem 2.2,

$$\left. \frac{d\pi_Z}{d\pi} \right|_{\mathcal{N}_t} = \Lambda_{y(Z, t)}.$$

From the form of y , it is clear that $\Lambda_{y(Z,t)} > 0$ for all $t \geq 0$ and that

$$\left(\Lambda_{y(Z,t)}, t \geq 0 \right)$$

is uniformly integrable. Point (i) of Theorem 3.7 then entails that \mathbf{Y} is left invertible and then invertible. \square

The theorem reads as follows:

Theorem 4.2 — Variational representation of the entropy. *Let $f : \mathfrak{N} \rightarrow \mathbf{R}$ such that*

$$\mathbf{E}_\pi \left[|f|(1 + e^f) \right] < \infty.$$

Then,

$$\log \mathbf{E}_\pi \left[e^f \right] = \sup_{y \in \mathcal{D}_m^{++}} \left(\mathbf{E}_\pi \left[f(N \circ y^*) \right] - \mathbf{E}_\pi \left[\int_0^\infty m(\dot{y}^*(s) - 1) ds \right] \right).$$

Proof. The duality between the relative entropy and the logarithmic Laplace transform says that

$$\log \mathbf{E}_\pi \left[e^f \right] = \sup_{\mu \in \mathfrak{M}^1(\mathfrak{N})} \left(\int_{\mathfrak{N}} f d\mu - H(\mu | \pi) \right) \quad (55)$$

where $\mathfrak{M}^1(\mathfrak{N})$ is the set of probability measures on \mathfrak{N} which are absolutely continuous with respect to π on \mathcal{N}_∞ . Furthermore, the supremum is attained at the measure μ_f whose π -density is given by

$$\frac{d\mu_f}{d\pi} = \frac{e^f}{\mathbf{E}_\pi \left[e^f \right]}.$$

In view of (55), we evidently have

$$\log \mathbf{E}_\pi \left[e^f \right] \geq \sup_{\mu \in \mathfrak{M}_m^1(\mathfrak{N})} \left(\int_{\mathfrak{N}} f d\mu - H(\mu | \pi) \right).$$

According to Lemma 4.1, for $y \in \mathcal{D}_m^{++}$

$$\int_{\mathfrak{N}} f d\mu - H(\mathbf{Y}^\# \pi | \pi) \geq \mathbf{E}_\pi \left[f(N \circ y^*) \right] - \mathbf{E}_\pi \left[\int_0^\infty m(\dot{y}^*(s) - 1) ds \right].$$

Since $\mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi) \subset \mathcal{D}_m^{++}$, the entropic criterion implies that

$$\begin{aligned} \sup_{\mu \in \mathfrak{M}^1(\mathfrak{N})} \int_{\mathfrak{N}} f d\mu - H(\mu | \pi) &\geq \sup_{y \in \mathcal{D}_m^{++}} \mathbf{E}_\pi \left[f(N \circ y^*) \right] - \mathbf{E}_\pi \left[\int_0^\infty m(\dot{y}^*(s) - 1) ds \right] \\ &\geq \sup_{y \in \mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)} \mathbf{E}_\pi \left[f(N \circ y^*) \right] - H(\mathbf{Y}^\# \pi | \pi). \end{aligned}$$

It remains to prove that we can find $(\dot{y}_n, n \geq 1)$, a sequence of elements of $\mathcal{D}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)$ such that

$$\begin{aligned} \int_{\mathfrak{N}} f d(\mathbf{Y}_n^\# \pi) &\xrightarrow{n \rightarrow \infty} \int_{\mathfrak{N}} f d\mu_f \\ H(\mathbf{Y}_n^\# \pi | \pi) &\xrightarrow{n \rightarrow \infty} H(\mu | \pi) \end{aligned}$$

to conclude. This is the object of the next theorem. \square

Theorem 4.3 Let ν be a probability measure on \mathfrak{N} absolutely continuous with respect to π and

$$L = \frac{d\nu}{d\pi}.$$

Assume that $L \log L \in L^1(\pi)$ and $\log L \in L^r(\pi)$ for some $r > 1$.

Then, there exists $(\dot{y}_n, n \geq 1)$ a sequence of elements of $\mathcal{P}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)$ such that

$$L_n \log L_n \xrightarrow[n \rightarrow \infty]{L^1(\pi)} L \log L \quad (56)$$

and

$$L_n \log L \xrightarrow[n \rightarrow \infty]{L^1(\pi)} L \log L \quad (57)$$

where

$$L_n = \frac{d\mathbf{Y}_n^\# \pi}{d\pi}.$$

Proof. We first show that we can suppose L lower and upper bounded. Consider

$$\Phi_n = (L \wedge n) \vee \frac{1}{n}.$$

We have

$$|\Phi_n| \leq L + 1,$$

hence by dominated convergence, Φ_n converges in $L^1(\pi)$ to L and in particular,

$$\mathbf{E}_\pi[\Phi_n] \xrightarrow{n \rightarrow \infty} \mathbf{E}_\pi[L] = 1.$$

Let

$$L_n = \frac{\Phi_n}{\mathbf{E}_\pi[\Phi_n]}.$$

For any $\alpha \in (0, 1)$, for n sufficiently large, $\mathbf{E}_\pi[\Phi_n] \geq \alpha$. Moreover, for $x \geq 0$, we have

$$|x \log(x)| \leq \frac{1}{e} + \left| \frac{x}{\alpha} \log\left(\frac{x}{\alpha}\right) \right| \mathbf{1}_{x \geq \alpha}.$$

Hence,

$$|L_n \log L_n| \leq \frac{1}{e} + \left| \frac{L}{\alpha} \log\left(\frac{L}{\alpha}\right) \right|.$$

By dominated convergence again, $L_n \log L_n$ converges to $L \log L$ in $L^1(\pi)$. Similarly,

$$|L_n \log L| \leq \left| \frac{L}{\alpha} \log L \right|$$

and $L_n \log L$ converges to $L \log L$ in $L^1(\pi)$.

Assume now that L is lower and upper bounded by respectively m and M . We know that there exists $\dot{y} \in \mathcal{P}^{++}(\mathcal{N}, \pi)$ such that

$$L = \Lambda_{\dot{y}}.$$

Set

$$L_n = \Lambda_{1+(\dot{y}^n-1)} = \mathbf{E}_\pi [L | \mathcal{N}_n].$$

Since L is bounded, it is clear that (56) and (57) hold. We can then assume that there exists $T > 0$ such that $\dot{y}(N, s) = 1$ for $s \geq T$.

Moreover, from the Malliavin calculus for Poisson process, we know (see [9]) that

$$\dot{y}(N, s) = \frac{\mathbf{E}_\pi [D_s L | \mathcal{N}_s]}{\mathbf{E}_\pi [L | \mathcal{N}_s]}$$

where $D_s F(N) = F(N + \epsilon_s) - F(N)$. We then have

$$0 \leq \dot{y}(N, s) \leq \frac{M}{m}.$$

Consider $\dot{y}_n = \dot{y} \vee n^{-1}$, it is straightforward that (56) and (57) hold.

Finally, assume that \dot{y} is lower and upper bounded on some interval $[0, T]$ and equal to 1 above T . Set

$$\begin{aligned} \dot{y}_n(s) &= 0 \text{ if } s \in [0, T/n) \\ \dot{y}_n(s) &= n \int_{(i-1)/n}^{i/n} \dot{y}(N, s) ds \text{ if } s \in [iT/n, (i+1)T/n) \end{aligned}$$

for $i \in \{1, \dots, n-1\}$. We see that \dot{y}_n belongs to $\mathcal{P}_{\infty, \text{pc}}^{++}(\mathcal{N}, \pi)$. We know (see [22]) that \dot{y}_n converges in $L^2(\mathfrak{N} \times [0, T], \pi \otimes ds)$ to \dot{y} . Moreover, it is easy to see that

$$\sup_n \mathbf{E}_\pi [\Lambda_{\dot{y}_n}^p] < \infty \text{ for any } p \geq 1.$$

Thus, (56) and (57) hold. □

5 Weak and strong g-Hawkes processes

This section does not directly utilize the prior results; rather, it is inspired by the analogy drawn between addressing a volatility-1 Brownian stochastic differential equation (SDE) and the formulation of a generalized Hawkes process. In the context of SDEs, three distinct types of solutions are recognized: strong solutions if for any given filtered probability space on which B is built, we can build a process X which satisfies (1); weak solutions if we have to specify the probability space; and martingale solutions which require that the expression

$$X(t) - \int_0^t b(X(s), s) ds$$

is a local martingale of square bracket ($t \mapsto t$). The interrelations among these various types of solutions have been well-documented in the literature and can be found in numerous textbooks, such as [23]. This culminates in the Yamada-Watanabe theorem, which asserts that weak existence and strong uniqueness together imply strong existence. In this work, we demonstrate that analogous definitions of these different types of solutions can be established for the construction of Hawkes processes, and we find a precise correspondence to the Yamada-Watanabe theorem.

Definition 5.1 A filtered probability space is a triplet $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is a space, equipped with a right-continuous filtration \mathcal{F} and a probability \mathbf{P} .

In what follows, we equip \mathfrak{N} with the minimum filtration:

$$\mathcal{N}_t = \sigma \{ \omega([0, s]), 0 \leq s \leq t \}.$$

The process $y : \mathfrak{N} \rightarrow \mathbf{R}$ is supposed to be \mathcal{N} -predictable and y is positive. We define the different notions of solution associated to the generalized Hawkes problem associated to y .

Definition 5.2 — g-Hawkes problem. Consider $(\Omega, \mathcal{F}, \mathbf{P})$ a filtered probability space. By a solution of g-H_y , we mean a couple of processes $Y = (Z, N)$ such that

1. With probability one, Z and N are point processes in the sense of Definition 2.1,
2. For any $t \geq 0$, the random variable $y(Z, t)$ is a \mathcal{F} stopping time,
3. The process $(N(t) - t, t \geq 0)$ is a \mathcal{F} -local martingale,
4. The processes Z and N satisfy, \mathbf{P} -a.s. for any $t \geq 0$,

$$Z(t) = N(y(Z, t)). \quad (58)$$

■ **Remark 4** Note that item 1. of Definition 5.2 and (58) imply that

$$y(\{T_1(Z), \dots, T_{q-1}(Z)\}, T_q(Z)) = T_q(N), \quad \forall q \geq 1. \quad (59)$$

However, (59) does not imply immediately (58) as it is not clear that $y(Z, t)$ is an \mathcal{N} stopping time and thus that the application $t \mapsto N(y(Z, t))$ is measurable. ■

Definition 5.3 — Weak and strong solutions. If we must specify the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ then the solution is said to be weak.

If we can find a solution for any filtered probability space on which we can construct a unit rate Poisson process, then the solution is said to be strong.

Definition 5.4 — Martingale problem. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be given and Z a point process. If the following conditions are satisfied:

- i) $y^*(Z, t)$ is \mathcal{F} -adapted,
- ii) the process

$$t \mapsto Z(y^*(Z, t)) - t$$

is a \mathcal{F} -local martingale,

we then say that (Z, \mathcal{F}) satisfies the g-Hawkes martingale problem, denoted by g-H_y^m .

The first theorem is the exact analog of what happens for stochastic differential equations driven by a Brownian motion.

Theorem 5.1 — Equivalence of weak and martingale solutions. *Let $\dot{y} \in \mathcal{D}^{++}(\mathcal{F})$. There exists a weak solution to $g\text{-H}_y$ if and only if there exists a solution to $g\text{-H}_y^m$.*

Proof. Assume that there exists $(\Omega, \mathcal{F}, \mathbf{P})$ and $Y = (Z, N)$ a solution to $g\text{-H}_y$. We assumed that $\dot{y} > 0$, thus

$$(y^*(Z, t) \leq s) = (y(Z, s) \geq t) \in \mathcal{F}_t,$$

since $y(Z, t)$ is an \mathcal{F} -stopping time. Thus the process $(y^*(Z, t), t \geq 0)$ is \mathcal{F} -adapted. Moreover, as $y^*(Z, \cdot)$ is an homeomorphism on \mathbf{R}^+ , we have, \mathbf{P} -a.s.,

$$Z(y^*(Z, t)) = N(t), \quad \forall t \geq 0,$$

and then

$$t \longmapsto Z(y^*(Z, t))$$

is an \mathcal{F} -Poisson process of intensity 1, i.e. point ii) holds and (Z, \mathcal{F}) solves $g\text{-H}_y^m$.

Conversely, if $(\Omega, \mathcal{F}, \mathbf{P}, Z)$ solves $g\text{-H}_y^m$, this means that N defined by

$$N(t) := Z(y^*(Z, t))$$

is a unit rate Poisson process with respect to the filtration \mathcal{F} and taking the inverse of $y^*(Z, \cdot)$, we have

$$Z(t) = N(y(Z, t)),$$

thus (58) is satisfied. Moreover, from i) we deduce that

$$(y(Z, t) \leq s) = (y^*(Z, s) \geq t) \in \mathcal{F}_s.$$

Hence Point 2 of Definition 5.2 is satisfied and we can then say that (Z, N) satisfies $g\text{-H}_y$ on $(\Omega, \mathcal{F}, \mathbf{P})$ where N is defined by (5). \square

Definition 5.5 — Pathwise uniqueness. We say that path-wise uniqueness holds for $g\text{-H}_y$ whenever for any two solutions $Y = (Z, N)$ and $Y' = (Z', N')$ defined on the same filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we have

$$Z = Z', \quad N = N'$$

up to \mathbf{P} -indistinguishability.

Definition 5.6 — Weak Uniqueness. We say that weak uniqueness holds for $g\text{-H}_y$ whenever for any two solutions $Y = (Z, N)$ and $Y' = (Z', N')$, possibly defined on different probability spaces, the law of Z and Z' on the space $(\mathfrak{N}, \mathcal{N}_\infty)$ do coincide.

For any $k \geq 1$, on \mathfrak{N}^k , we consider the σ -field $\mathcal{N}^{\otimes(k)}$ defined by

$$\mathcal{N}^{\otimes(k)} = \otimes_{i=1}^k \mathcal{N}^i \text{ where } \mathcal{N}^i = \sigma(\eta_i(s), s \geq 0), \text{ for } i = 1, \dots, k.$$

For any $t \geq 0$, we introduce the sub- σ field

$$\mathcal{N}_t^{\otimes(k)} = \otimes_{i=1}^k \mathcal{N}_t^i.$$

For any solution Y of $g\text{-H}_y$, let μ be the law of Y on $(\mathfrak{N}^2, \mathcal{N}^{\otimes(2)})$, i.e. $\mu = Y^\# \mathbf{P}$. Let $\mu_{\eta_2}(d\eta_1)$ be the regular conditional distribution of $\mu(d\eta_1, d\eta_2)$ given η_2 :

1. For each η_2 , $\mu_{\eta_2}(\mathrm{d}\eta_1)$ is a probability measure on $(\mathfrak{N}, \mathcal{N}^1)$,
2. For each $B \in \mathcal{N}^1$, $\mu_{\eta_2}(B)$ is \mathcal{N}^2 -measurable in η_2 ,
3. For any $B \in \mathcal{N}^1$, $B' \in \mathcal{N}^2$,

$$\mu(B \times B') = \int_{B'} \mu_{\eta_2}(B) \mathrm{d}\pi(\eta_2).$$

Let $\mu_{\eta_2}^t$ be the regular conditional distribution of μ given \mathcal{N}_t :

1. For each $\eta_2 \in \mathfrak{N}$, $\mu_{\eta_2}^t$ is a measure on \mathfrak{N} ,
2. for each $B \in \mathcal{N}$, the random variable $\mu_{\eta_2}^t(B)$ is $\mathcal{N}_t / \mathcal{B}(\mathbf{R}^+)$ -measurable,
3. for any $B' \in \mathcal{N}_t$,

$$\mu(B \times B') = \int_{B'} \mu_{\eta_2}^t(B) \mathrm{d}\pi(\eta_2).$$

Lemma 5.2 For $B \in \mathcal{N}_{y^*(\eta_1, t)}^1$, the map $\eta_2 \mapsto \mu_{\eta_2}(B)$ is $\mathcal{N}_t^2 / \mathcal{B}(\mathbf{R}^+)$ -measurable.

Proof. We first prove that for $B \in \mathcal{N}_{y^*(\eta_1, t)}^1$, there exists θ measurable, such that

$$\mathbf{1}_B(\eta_1) = \theta(\eta_2^t), \mu\text{-a.s.} \quad (60)$$

or equivalently, $\mathbf{1}_B(\eta_1) \in \mathcal{N}_t^2$. According to [15], we know that $\mathcal{N}_{y^*(\eta_1, t)}^1$ is generated by sets of the form

$$\bigcap_{j=0}^q \left(T_j(\eta_1) \leq a_j \right) \cap \left(T_q(\eta_1) < y^*(\eta_1, t) \leq T_{q+1}(\eta_1) \right) \quad (61)$$

for some $a_j \in \mathbf{R}^+$ and some $q \in \mathbf{N}$. Recall that (58) implies that

$$\mu\left(\eta_1(t) = \eta_2(y(\eta_1, t)), \forall t \geq 0 \right) = 1.$$

This means that

$$y(\eta_1, T_k(\eta_1)) = T_k(\eta_2) \iff T_k(\eta_1) = y^*(\eta_1, T_k(\eta_2)).$$

Hence for a set B of the form (61), we have

$$\begin{aligned} B &= \bigcap_{j=0}^q \left(y^*(\eta_1, T_j(\eta_2)) \leq a_j \right) \cap \left(T_q(\eta_2) < t \leq T_{q+1}(\eta_2) \right) \\ &= \bigcap_{j=0}^q \left(T_j(\eta_2) \leq y(\eta_1, a_j) \right) \cap \left(T_q(\eta_2) < t \leq T_{q+1}(\eta_2) \right) \end{aligned}$$

By definition of the g-Hawkes problem, $y(\eta_1, a)$ is a \mathcal{N}^2 -stopping time hence for any $j \leq q$,

$$\left(T_j(\eta_2) \leq y(\eta_1, a_j) \right) \in \mathcal{N}_{T_j(\eta_2) \wedge y(\eta_1, a_j)}^2 \subset \mathcal{N}_{T_j(\eta_2)}^2 \subset \mathcal{N}_{T_q(\eta_2)}^2.$$

Hence $B \in \mathcal{N}_t^2$.

Let $B \in \mathcal{N}_{y^*(\eta_1, t)}^1$. For the second step of the proof, we have to prove that for $F : \mathfrak{N} \rightarrow \mathbf{R}$ measurable and bounded

$$\int_{\mathfrak{N} \times \mathfrak{N}} F(\eta_2) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) = \int_{\mathfrak{N} \times \mathfrak{N}} F(\eta_2) \mu_{\eta_2}^t(B) \mu(d\eta_1, d\eta_2).$$

Since η_2 is a Poisson process, for any F bounded, the martingale representation theorem valid for Poisson processes says that there exists u_F which is \mathcal{N} -adapted such that

$$\mathbf{E} \left[\int_0^\infty u_F(\eta_2, s)^2 ds \right] < \infty$$

and

$$F(\eta_2) = \mathbf{E}[F] + \int_0^\infty u_F(\eta_2, s) d\tilde{\eta}_2(s)$$

where $\tilde{\eta}_2(t) = \eta_2(t) - t$. The process

$$(\eta_2, t) \longmapsto \int_0^t u_F(\eta_2, s) d\tilde{\eta}_2(s)$$

is a square integrable martingale with respect to the filtration \mathcal{N} and thus it is also a square martingale with respect to the filtration $\mathcal{N}^{\otimes(2)}$. Now then we have

$$\begin{aligned} \int_{\mathfrak{N} \times \mathfrak{N}} F(\eta_2) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) &= \mathbf{E}[F] \int_{\mathfrak{N} \times \mathfrak{N}} \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) \\ &+ \int_{\mathfrak{N} \times \mathfrak{N}} \left(\int_t^\infty u_F(\eta_2, s) d\tilde{\eta}_2(s) \right) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2) \\ &+ \int_{\mathfrak{N} \times \mathfrak{N}} \left(\int_0^t u_F(\eta, s) d\tilde{\eta}(s) \right) \mathbf{1}_B(\eta_1) \mu(d\eta_1, d\eta_2). \end{aligned}$$

By the first part of the proof, $\mathbf{1}_B(\eta_1) = \theta(\eta_2^t) \mu$ -a.s. Hence, by the martingale property of the stochastic integral, the median term is null. We thus get

$$\begin{aligned} \int_{\mathfrak{N} \times \mathfrak{N}} F(\eta) \mathbf{1}_B(z) \mu(dz, d\eta) &= \int_{\mathfrak{N}} \left[\mathbf{E}[F] + \left(\int_0^t u_F(\eta, s) d\tilde{\eta}(s) \right) \right] \mu_{\eta}^t(B) d\pi(\eta) \\ &= \int_{\mathfrak{N}} F(\eta) \mu_{\eta}^t(B) d\pi(\eta) \end{aligned}$$

by the same kind of reasoning. □

Theorem 5.3 — Yamada-Watanabe. *With the same notations as above. Consider the g - H_y problem as in Definition 5.2. The following two properties hold:*

1. *Pathwise uniqueness implies weak uniqueness.*
2. *Moreover, if there exists a solution $Y = (Z, N)$ of g - H_y on some $(\Omega, \mathcal{F}, \mathbf{P})$ and pathwise uniqueness holds then there exists $F : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $Z = F(N)$ \mathbf{P} -a.s. Fur-*

thermore, the map F is $\mathcal{N}_t / \mathcal{Z}_{y^*(Z,t)}$ measurable, i.e. for any $t \geq 0$,

$$\mathcal{Z}_{y^*(Z,t)} = \sigma \{F(N(s)), s \leq t\}.$$

Proof. Let $Y = (Z, N)$ and $Y' = (Z', N')$ two solutions of $g\text{-H}_y$, which are possibly defined on different probability spaces, and μ and μ' there respective distribution on $(\mathfrak{N}^2; \mathcal{N}_\infty^{\otimes(2)})$. On $(\mathfrak{N}^3, \mathcal{N}_\infty^{\otimes(3)})$, we define the probability measure

$$\nu(d\eta_1, d\eta_2, d\eta_3) = \mu_{\eta_3}(d\eta_1) \mu'_{\eta_3}(d\eta_2) d\pi(\eta_3).$$

We are going to prove that η_3 is a ν Poisson process. For, for $i = 1, 2, 3$, let F_i a bounded function, $\mathcal{N}_{y^*(\eta_i,s)}^i / \mathcal{B}(\mathbf{R})$ measurable for $i = 1, 2$ and $\mathcal{N}_s^3 / \mathcal{B}(\mathbf{R})$ measurable for $i = 3$. We have

$$\begin{aligned} & \int_{\mathfrak{N}^3} F_1(\eta_1) F_2(\eta_2) F_3(\eta_3) (\tilde{\eta}_3(t) - \tilde{\eta}_3(s)) \nu(d\eta_1, d\eta_2, d\eta_3) \\ &= \int_{\mathfrak{N}} \left(\int_{\mathfrak{N}} F_1(\eta_1) \mu_{\eta_3}(d\eta_1) \right) \left(\int_{\mathfrak{N}} F_2(\eta_2) \mu'_{\eta_3}(d\eta_2) \right) (\tilde{\eta}_3(t) - \tilde{\eta}_3(s)) d\pi(\eta_3). \end{aligned} \quad (62)$$

where $\tilde{\eta}_3(t) = \eta_3(t) - t$. According to Lemma 5.2, the random variables

$$\int_{\mathfrak{N}} F_1(\eta_1) \mu_{\eta_3}(d\eta_1) \text{ and } \int_{\mathfrak{N}} F_2(\eta_2) \mu'_{\eta_3}(d\eta_2)$$

are \mathcal{N}_s^3 measurable hence $\mathcal{N}_s^{\otimes(3)}$ -measurable. It follows that the right-hand-side of (62) is equal to zero and that η_3 is a $(\nu, (\mathcal{N}_{y^*(\eta_1,t)}^1 \otimes \mathcal{N}_{y^*(\eta_2,t)}^2 \otimes \mathcal{N}_t^3, t \geq 0))$ unit rate Poisson process.

We thus have two solutions (η_1, η_3) and (η_2, η_3) on the same probability space. The path-wise uniqueness then implies that $\eta_1 = \eta_2$, ν -a.s. Thus, $\mu(d\eta_1, d\eta_3) = \mu'(d\eta_2, d\eta_3)$ and the uniqueness in law holds.

In view of Lemma 2.5 and Corollary 3.5, the Doob Lemma says that there exist two functions $F_i, i = 1, 2$ respectively measurable from $(\mathfrak{N}, \mathcal{N}_\infty^3)$ to $(\mathfrak{N}, \mathcal{N}_{y^*(\eta_i,\infty)}^i) = (\mathfrak{N}, \mathcal{N}_\infty^i), i = 1, 2$ such that $\eta_1^{y^*(\eta_1,t)} = F_1(\eta_3^t)$ and $\eta_2^{y^*(\eta_2,t)} = F_2(\eta_3^t)$ for any $t \geq 0$. Furthermore, for π -almost all η_3 ,

$$\mu_{\eta_3} \otimes \mu_{\eta_3}(\eta_1 = \eta_2) = 1$$

and this implies that $F_1 = F_2$, i.e. there exists F such that $\eta_1 = \eta_2 = F(\eta_3)$. \square

Theorem 5.4 *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and N a unit Poisson process such that $\mathcal{F} = \mathcal{N}$. We denote by π the law of N . For any $\dot{y} \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$, there exists a (weak) solution to $g\text{-H}_y$.*

Proof. In view of Theorem 2.2, π_y is locally absolutely continuous with respect to π and the quasi-invariance theorem says that the process Y defined by

$$Y(t) := N(y^*(N, t)),$$

is a $(\pi_y, \mathcal{F}^{y^*})$ unit Poisson process. Thus, we have

$$N(t) = Y(y(N, t))$$

and $y(N, t)$ is a \mathcal{F}^{y^*} -stopping time, hence $(N, Y, \mathcal{F}^{y^*})$ solves $g\text{-H}_y^w$. \square

Theorem 5.5 Consider $(\Omega, \mathcal{F}, \mathbf{P})$ be a filtered probability space and N a unit Poisson process. Let $y \in \mathcal{D}_2^{++}(\mathcal{N}, \pi_y)$ such that for any $\omega, \eta \in \mathfrak{N}$, for any $t \geq 0$

$$|y(\omega, t) - y(\eta, t)| \leq \int_0^t |\phi(t-s)| d|\omega - \eta|(s) \quad (63)$$

where $|\omega - \eta|$ is the point process defined by

$$|\omega - \eta|(t) = |\omega(t) - \eta(t)|$$

and ϕ is such that

$$\int_0^\infty \phi(s) ds < 1.$$

Then, there exists a unique solution to $g\text{-H}_y^s$.

■ **Remark 5** The condition (63) is the condition used in [17, 20] to ensure existence and uniqueness of the solution of the (weak) g -Hawkes problem. ■

Proof. The weak existence is guaranteed by Theorem 5.4. It remains to show the strong uniqueness and to conclude with Theorem 5.3. If X and Y are two solutions of $g\text{-H}_y$ on the same probability space, we have

$$X(t) = N(y(X, t)) \text{ and } Y(t) = N(y(Y, t)).$$

Since they both are point processes, we can speak of their jump times $(T_q(X), q \geq 1)$ and $(T_q(Y), q \geq 1)$. We have

$$y(\emptyset, T_1(X)) = T_1(N) \text{ and } y(\emptyset, T_1(Y)) = T_1(N).$$

Since $y(N, \cdot)$ is an homeomorphism, $T_1(X) = T_1(Y)$. Suppose proved that

$$T_j(X) = T_j(Y) \text{ for } j = 1, \dots, q-1.$$

Since y is predictable

$$y\left(\sum_{j=1}^{q-1} \epsilon_{T_j(X)}, T_q(X)\right) = T_q(N) = y\left(\sum_{j=1}^{q-1} \epsilon_{T_j(Y)}, T_q(Y)\right).$$

Hence $T_q(X) = T_q(Y)$ and then $X = Y$. Thus the strong uniqueness holds. \square

■ **Remark 6** Actually, we have used in this proof the construction inherited from Algorithm 7.4.III of [7] to simulate a g-Hawkes process. In view of (37) and (19), $T_1(Z)$ is the solution of the equation

$$y(\emptyset, T_1(Z)) = T_1(N).$$

In turn, $T_2(Z)$ solves

$$y(\epsilon_{T_1(Z)}, T_2(Z)) = T_2(N).$$

More generally, $T_k(Z)$ is given by

$$y\left(\sum_{j=1}^{k-1} \epsilon_{T_j(Z)}, T_k(Z)\right) = T_k(N).$$

The usual algorithm which is based on the thinning of a Poisson measure gives the sample path of Z up to a given time and suffers from the rejection of a possibly large number of points. This algorithm yields the number of jumps we desire and may suffer from numerical instability in the root findings. ■

We can now retrieve the classical existence result for classical Hawkes processes [13, 20].

Corollary 5.6 *Let \dot{y} be a predictable process which satisfies the following hypothesis*

$$\dot{y}(N, t) = \varphi\left(\alpha + \int_0^t h(t-s) dN(s)\right)$$

where $\alpha > 0$, φ is a Lebesgue a.s. positive Lipschitz function and h is a non-negative measurable function such that

$$\|\varphi\|_{Lip} \int_0^\infty h(t) dt < 1. \quad (64)$$

Then, there exists a unique strong solution to g- H_y .

Proof. Let $T > 0$ and

$$\dot{y}_T(t) = \begin{cases} \dot{y}(N, s) & \text{if } s \leq T \\ 1 & \text{if } s > T. \end{cases}$$

We first prove that there exists a solution to \mathfrak{H}_{y_T} . We have to prove that

$$\pi_y\left(\int_0^T \left(1 - \sqrt{\dot{y}(N, s)}\right)^2 ds < \infty\right) = 1. \quad (65)$$

It is easily seen that for all $x \geq -1$, we have

$$\left(\sqrt{1+x} - 1\right)^2 \leq x^2 \mathbf{1}_{\{|x| \leq 1\}} + x \mathbf{1}_{\{|x| > 1\}}.$$

Since φ is supposed to be Lipschitz continuous, we get

$$\begin{aligned} \int_0^T \left(1 - \sqrt{\dot{y}(N, s)}\right)^2 ds &\leq T + \int_0^T |\dot{y}(N, s) - 1| ds \\ &\leq (1 + |\varphi(0) - 1| + \alpha \|\varphi\|_{\text{Lip}}) T + \|\varphi\|_{\text{Lip}} \int_0^T \int_0^s h(s-u) dN(u) ds. \end{aligned}$$

Thus, proving (65) amounts to prove that

$$\pi_y \left(\int_0^T \int_0^s h(s-u) dN(u) ds < \infty \right) = 1.$$

Denote by H the first quadrature of h :

$$H(t) = \int_0^t h(s) ds.$$

By Fubini's Theorem, we have

$$\int_0^T \int_0^s h(s-u) dN(u) ds = \int_0^T H(T-s) dN(s) \leq H(T)N(T).$$

The same argument shows that

$$\mathbf{E}_{\pi_y} [N(T)] \leq cT + \|\varphi\|_{\text{Lip}} H(T) \mathbf{E}_{\pi_y} [N(T)].$$

In view of (64), this induces that $\mathbf{E}_{\pi_y} [N(T)]$ is finite and thus that (65) holds. It is clear that \dot{y} so defined satisfies (63) and then, Theorem 5.5 ensures the existence of the Hawkes process associated to y on $[0, T]$ for any $T > 0$. We denote this process by Y_T . Since y is predictable, for all $r \leq T \leq S$,

$$\begin{aligned} Y_T(r) &= N(y_T(Y_T, r)) \\ &= N(y_S(Y_T, r)). \end{aligned}$$

By uniqueness, this means that Y_T and Y_S coincide on $[0, T]$. We can then define

$$Y(r) = Y_r(r),$$

which satisfies

$$Y(r) = N(y_r(Y(r), r)) = N(y(Y(r), r))$$

by the very definition of \dot{y}_r . We have thus proved the existence of a strong solution to the Hawkes problem associated to y . The uniqueness comes from the Lipschitz condition (63) as in the proof of Theorem 5.5. \square

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