

The absorptive nature of the scattering coefficient in the stress-energy tensor formalism for room acoustics

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ABSTRACT

In the stress-energy tensor formalism, the symmetry between absorption and scattering coefficients, as proven by measurements combined with simulations, is counter-intuitive. By introducing the wall admittance, we show that the scattering coefficient is partly created by the real part of the wall admittance combined with the active intensity, that is, is partly due to absorption. However, it also depends on the imaginary part of the wall admittance in combination with the reactive intensity, which confers it genuine scattering properties. In the case of plane waves impinging on planar boundary, the admittance formalism shows that reactive intensity vanishes in directions parallel to the wall; when the source is at finite distance from the wall, a residual reactive intensity subsists. However, for curved boundaries, the velocity in directions parallel to the wall is no longer proportional to the pressure, and scattering occurs.

Keywords: Energy density, Sound intensity, Absorption coefficient, Scattering coefficient, Wall admittance

1 INTRODUCTION

The stress-energy tensor formalism, introduced into Acoustics by Morse and Ingard [1], was further developed by Dujourdy *et al.* [2, 3] to generalize the diffusion equation formalism of Ollendorf and Picaut [4, 5]: instead of a gradient type relationship between sound intensity and total energy, it introduces instead the conservation equation for intensity. After integration on the cross-section, it yields both absorption and scattering on the boundaries. Dujourdy *et al.* stressed the symmetrical role played by absorption and scattering coefficients, which was confirmed by Meacham *et al.* [6]. The present papers aims at proving this symmetry and the absorbing nature of the scattering coefficient.

2 THE BASIC EQUATIONS

Dujourdy *et al.* [2, 3], following [1], have shown that the wave equation can be extended by a set of conservation equations that reduces to the conservation of the stress-energy tensor \underline{T} :

$$\vec{\nabla} \cdot \underline{T} = 0 \quad (1)$$

with

$$\underline{T} = \begin{pmatrix} E_{tt} & E_{tx} & E_{ty} & E_{tz} \\ E_{tx} & E_{xx} & E_{xy} & E_{xz} \\ E_{ty} & E_{xy} & E_{yy} & E_{yz} \\ E_{tz} & E_{xz} & E_{yz} & E_{zz} \end{pmatrix} \quad (2)$$

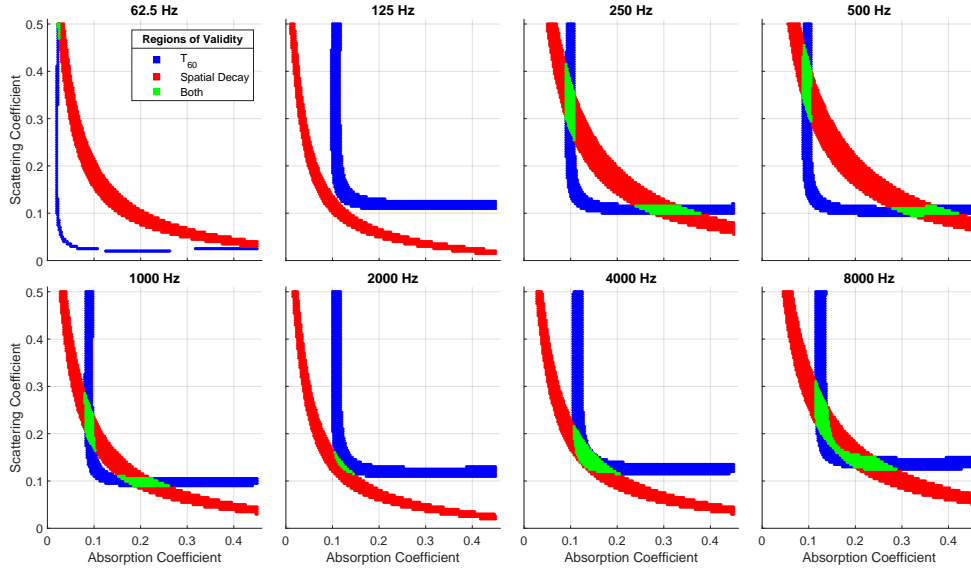


Figure 1. Relevant combinations of absorption and scattering coefficients leading to measured reverberation times and spatial decays in a hallway (from [6]).

In terms of the velocity potential ψ , the elements of the stress-energy tensor are given by:

$$\begin{aligned}
 E_{tt} &= \frac{\rho}{2} \left(\frac{1}{c^2} |\partial_t \psi|^2 + |\partial_x \psi|^2 + |\partial_y \psi|^2 + |\partial_z \psi|^2 \right) & E_{tx} &= -\frac{\rho}{c} \Re(\partial_t \psi \partial_x \psi^*) & E_{xy} &= \rho \Re(\partial_x \psi \partial_y \psi^*) \\
 E_{xx} &= \frac{\rho}{2} \left(\frac{1}{c^2} |\partial_t \psi|^2 + |\partial_x \psi|^2 - |\partial_y \psi|^2 - |\partial_z \psi|^2 \right) & E_{ty} &= -\frac{\rho}{c} \Re(\partial_t \psi \partial_y \psi^*) & E_{xz} &= \rho \Re(\partial_x \psi \partial_z \psi^*) \\
 E_{yy} &= \frac{\rho}{2} \left(\frac{1}{c^2} |\partial_t \psi|^2 - |\partial_x \psi|^2 + |\partial_y \psi|^2 - |\partial_z \psi|^2 \right) & E_{tz} &= -\frac{\rho}{c} \Re(\partial_t \psi \partial_z \psi^*) & E_{yz} &= \rho \Re(\partial_y \psi \partial_z \psi^*) \\
 E_{zz} &= \frac{\rho}{2} \left(\frac{1}{c^2} |\partial_t \psi|^2 - |\partial_x \psi|^2 - |\partial_y \psi|^2 + |\partial_z \psi|^2 \right)
 \end{aligned} \quad (3)$$

where ρ is the density of air and c the speed of sound.

For long enclosures, Dujourdy *et al.* [2] reduced the conservation of the stress-energy tensor (eq. 1) to the telegraphers equation:

$$\frac{1}{c^2} \partial_{tt} E - \partial_{xx} E + \frac{A+D}{\lambda c} \partial_t E + \frac{AD}{\lambda^2} E = 0 \quad (4)$$

where E is the total energy, λ is the mean free path of the enclosure, and A and D are respectively modified absorption and scattering coefficients at the boundaries. A similar equation exists for flat spaces [3].

Eq. (4) is symmetrical with respect to the absorption and scattering coefficients, as was confirmed by Meacham *et al.* [6] in Fig. 1, where the measured reverberation times and spatial decays correspond to narrow stripes of values for A and D , which do not always overlap. Note the symmetry of the figure with respect to absorption and diffraction coefficients.

3 WALL ADMITTANCE

Wall admittance introduces a relation between the normal derivative and the time derivative of the velocity potential on the wall:

$$-\partial_n \psi = \frac{\beta}{c} \partial_t \psi = \frac{1}{c\zeta} \partial_t \psi \quad (5)$$

with \vec{n} the exterior normal to the wall, ζ the reduced impedance of the wall, and $\beta = \frac{1}{\zeta}$ the reduced admittance. Let us then consider a flat enclosure with vertical coordinate z . We can express the elements E_z of the stress-

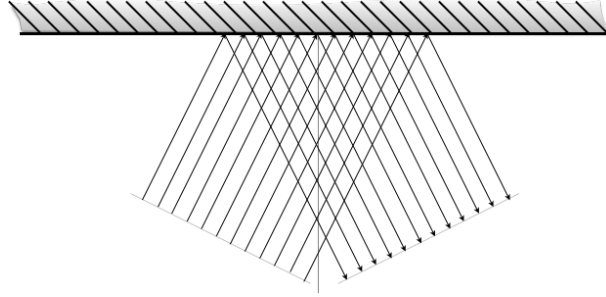


Figure 2. Plane wave reflecting on flat boundary.

energy tensor, where \cdot takes the values t , x , or y . We thus obtain on the "ceiling":

$$\begin{aligned}
 E_{tz} &= \frac{\rho}{c^2} \Re(\partial_t \psi \beta^* \partial_t \psi^*) = \frac{\rho}{c^2} \Re(\beta) |\partial_t \psi|^2 \\
 E_{xz} &= -\frac{\rho}{c} \Re(\partial_x \psi \beta^* \partial_t \psi^*) = \Re(\beta) E_{tx} + \Im(\beta) \Im\left(\frac{\rho}{c} \partial_x \psi \partial_t \psi^*\right) \\
 E_{yz} &= -\frac{\rho}{c} \Re(\partial_y \psi \beta^* \partial_t \psi^*) = \Re(\beta) E_{ty} + \Im(\beta) \Im\left(\frac{\rho}{c} \partial_y \psi \partial_t \psi^*\right)
 \end{aligned} \tag{6}$$

and the same equations with opposite sign on the "floor".

Now, E_{tx} and E_{ty} are the components of the sound intensity in the x and y directions. The boundary conditions for E_{xz} and E_{yz} therefore involve not only the active intensity, but also the reactive intensity, which corresponds to local recirculation of energy. For monochromatic waves, we can further consider the phase angles ϕ_x and ϕ_y between the space derivatives of the velocity potential with respect to x and y respectively, and its time derivative. The last two equations (6) then reduce to:

$$\begin{aligned}
 E_{xz} &= [\Re(\beta) + \Im(\beta) \tan \phi_x] E_{tx} \\
 E_{yz} &= [\Re(\beta) + \Im(\beta) \tan \phi_y] E_{ty}
 \end{aligned} \tag{7}$$

and E_{tt} to:

$$\begin{aligned}
 E_{tt} &= \frac{\rho}{2} \left(\frac{1}{c^2} [1 + |\beta|^2] |\partial_t \psi|^2 + |\partial_x \psi|^2 + |\partial_y \psi|^2 \right) \\
 &= \left[\frac{\rho}{2c^2} (1 + \sin^2 \theta + |\beta|^2) \right] |\partial_t \psi|^2
 \end{aligned} \tag{8}$$

after introducing the angle of incidence θ of the wave

The absorption coefficient then takes the following expression:

$$\alpha(\theta) = \frac{E_{tz}}{E_{tt}} = \frac{2 \Re(\beta)}{1 + \sin^2 \theta + |\beta|^2} \tag{9}$$

and the scattering coefficients D_x and D_y :

$$\begin{aligned}
 D_x &= \frac{E_{xz}}{E_{tx}} = [\Re(\beta) + \Im(\beta) \tan \phi_x] \\
 D_y &= \frac{E_{yz}}{E_{ty}} = [\Re(\beta) + \Im(\beta) \tan \phi_y]
 \end{aligned} \tag{10}$$

4 BOUNDARY CONDITIONS

4.1 Plane wave on flat boundary

For a plane wave impinging on a flat boundary $z = 0$ (Fig. 2), the velocity potential takes the following form:

$$\psi(t, r) = \exp -i(\omega t - k_x x - k_y y - k_z z) + R \exp -i(\omega t - k_x x - k_y y + k_z z) \tag{11}$$

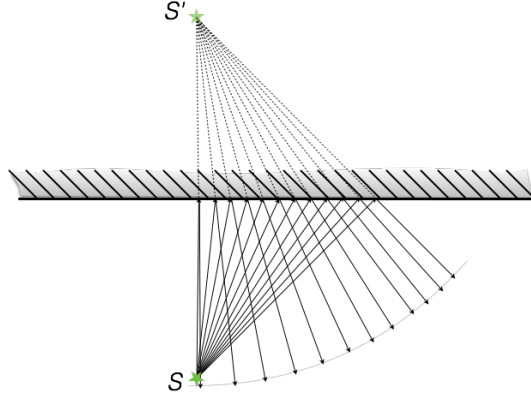


Figure 3. Source at finite distance from flat boundary; S is real source, and S' virtual image source.

where R is the complex reflection coefficient on the boundary. Its derivatives in the neighbourhood of the boundary are given by:

$$\begin{aligned}
 \partial_t \psi &= -i\omega \psi = -i\omega \exp -i(\omega t - k_x x - k_y y) [\exp ik_z z + R \exp -ik_z z] \\
 \partial_x \psi &= ik_x \psi = ik_x \exp -i(\omega t - k_x x - k_y y) [\exp ik_z z + R \exp -ik_z z] = -\frac{k_x}{\omega} \partial_t \psi \\
 \partial_y \psi &= ik_y \psi = ik_y \exp -i(\omega t - k_x x - k_y y) [\exp ik_z z + R \exp -ik_z z] = -\frac{k_y}{\omega} \partial_t \psi \\
 \partial_z \psi &= ik_z \exp -i(\omega t - k_x x - k_y y) [\exp ik_z z - R \exp -ik_z z] = -\frac{\beta}{c} \partial_t \psi
 \end{aligned} \tag{12}$$

As a consequence, on the boundary $z=0$, the phase angles ϕ_x and ϕ_y are both equal to 0, and eqs. (10) reduce to:

$$D_x = D_y = \Re(\beta) \tag{13}$$

that is, the diffraction coefficient is purely created by absorption.

4.2 Source at finite distance from flat boundary

For a spherical wave impinging on a flat boundary (Fig. 3), the velocity potential takes a different form:

$$\psi(t, r) = \frac{\exp -i(\omega t - kr)}{4\pi r} + R \frac{\exp -i(\omega t - kr')}{4\pi r'} \tag{14}$$

where R still is the complex reflection coefficient on the boundary, r the distance to the real source S , and r' the distance to the virtual image source S' (see Fig. 3). The derivatives in the neighbourhood of the boundary are now given by:

$$\begin{aligned}
 \partial_t \psi &= -i\omega \psi = -i\omega \frac{\exp -i\omega t}{4\pi} \left[\frac{\exp ikr}{r} + R \frac{\exp ikr'}{r'} \right] \\
 \partial_x \psi &= i \frac{\exp -i\omega t}{4\pi} \left[k_x \frac{\exp ikr}{r} \left(1 + \frac{i}{kr} \right) + R k'_x \frac{\exp ikr'}{r'} \left(1 + \frac{i}{kr'} \right) \right] \\
 \partial_y \psi &= i \frac{\exp -i\omega t}{4\pi} \left[k_y \frac{\exp ikr}{r} \left(1 + \frac{i}{kr} \right) + R k'_y \frac{\exp ikr'}{r'} \left(1 + \frac{i}{kr'} \right) \right] \\
 \partial_z \psi &= i \frac{\exp -i\omega t}{4\pi} \left[k_z \frac{\exp ikr}{r} \left(1 + \frac{i}{kr} \right) + R k'_z \frac{\exp ikr'}{r'} \left(1 + \frac{i}{kr'} \right) \right]
 \end{aligned} \tag{15}$$

where k_x , k_y and k_z are the components of the incident wave number k along the three coordinates, and k'_x , k'_y , and k'_z those of the reflected wave number k' . At the boundary, $r = r'$, $k_x = k'_x$, $k_y = k'_y$, but $k'_z = -k_z$; and the

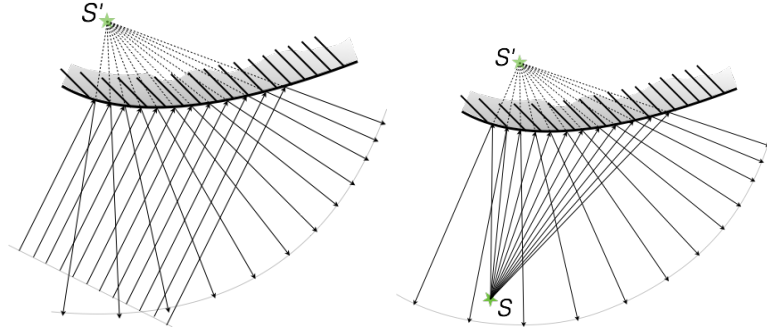


Figure 4. Reflection on curved boundary. Left: plane wave impinging; right: source at finite distance. S is real source, and S' virtual image source.

derivatives further reduce to:

$$\begin{aligned}
 \partial_t \psi &= -i\omega \frac{\exp-i(\omega t - kr)}{4\pi r} (1 + R) \\
 \partial_x \psi &= ik_x \frac{\exp-i(\omega t - kr)}{4\pi r} (1 + R) \left(1 + \frac{i}{kr}\right) = -\frac{k_x}{\omega} \left(1 + \frac{i}{kr}\right) \partial_t \psi \\
 \partial_y \psi &= ik_y \frac{\exp-i(\omega t - kr)}{4\pi r} (1 + R) \left(1 + \frac{i}{kr}\right) = -\frac{k_y}{\omega} \left(1 + \frac{i}{kr}\right) \partial_t \psi \\
 \partial_z \psi &= ik_z \frac{\exp-i(\omega t - kr)}{4\pi r} (1 - R) \left(1 + \frac{i}{kr}\right) = -\frac{\beta}{c} \partial_t \psi
 \end{aligned} \tag{16}$$

As a consequence, on the boundary, the phase angles ϕ_x and ϕ_y are both equal to $\arctan \frac{1}{kr}$, and eqs. (10) reduce to:

$$D_x = D_y = \Re(\beta) + \frac{\Im(\beta)}{kr} \tag{17}$$

The imaginary part of the admittance contributes to the diffraction coefficient.

4.3 Curved boundary

In the case of a curved boundary, the real and virtual sources are not located at the same distance from the boundary (Fig. 4). As a consequence, eqs. (15) are still valid in the vicinity of the boundary, provided that x , y and z are now considered as local coordinates at the boundary, the two first ones being parallel to the boundary and the last one perpendicular to it. However, they do not simplify into eq. (16) at the boundary, even though $k_x = k'_x$, $k_y = k'_y$, and $k'_z = -k_z$, because $r' \neq r$.

On the boundary, we now obtain from eqs. (15):

$$\begin{aligned}
 \partial_x \psi &= -\frac{k_x}{\omega} \left(1 + i \frac{\frac{\exp ikr}{kr^2} + R \frac{\exp ikr'}{kr'^2}}{\frac{\exp ikr}{r} + R \frac{\exp ikr'}{r'}} \right) \partial_t \psi \\
 \partial_y \psi &= -\frac{k_y}{\omega} \left(1 + i \frac{\frac{\exp ikr}{kr^2} + R \frac{\exp ikr'}{kr'^2}}{\frac{\exp ikr}{r} + R \frac{\exp ikr'}{r'}} \right) \partial_t \psi \\
 \partial_z \psi &= -\frac{k_z}{\omega} \left(\frac{\left[\frac{\exp ikr}{r} - R \frac{\exp ikr'}{r'} \right] + i \left[\frac{\exp ikr}{kr^2} - R \frac{\exp ikr'}{kr'^2} \right]}{\frac{\exp ikr}{r} + R \frac{\exp ikr'}{r'}} \right) \partial_t \psi = -\frac{\beta}{c} \partial_t \psi
 \end{aligned} \tag{18}$$

where

$$\frac{\left[\frac{\exp ikr}{kr^2} + R \frac{\exp ikr'}{kr'^2} \right]}{\left[\frac{\exp ikr}{r} + R \frac{\exp ikr'}{r'} \right]} = \frac{\left[\frac{1}{kr^2} + \frac{R'}{kr'^2} \right]}{\left[\frac{1}{r} + \frac{R'}{r'} \right]} \tag{19}$$

with $R' = R \exp ik(r' - r)$. We then obtain:

$$\tan \phi_x = \tan \phi_y = \frac{1}{k} \frac{\frac{1}{r^3} + \left(\frac{1}{r} + \frac{1}{r'}\right) \frac{\Re(R')}{rr'} + \frac{|R'|^2}{r^3}}{\frac{1}{r^2} + \frac{2\Re(R')}{rr'} + \frac{|R'|^2}{r^2} + \left(\frac{1}{r} - \frac{1}{r'}\right) \frac{\Im(R')}{kr'}} \quad (20)$$

As a consequence, the phase angles ϕ_x and ϕ_y are position dependent through the modified reflection coefficient R' . No simple equation subsists for eqs. (10) to reduce to.

Note that, in the case of a plane wave incident on a curved boundary (left pane of Fig. 4), eq. (20) reduce to:

$$\tan \phi_x = \tan \phi_y = \frac{1}{kr'^2} \frac{\Re(R') + \frac{|R'|^2}{r'}}{1 + \frac{2\Re(R')}{r'} + \frac{|R'|^2}{r'^2} - \frac{\Im(R')}{kr'^2}} \quad (21)$$

5 DISCUSSION

5.1 Absorption coefficient

The form for the absorption coefficient given by eq. (9) does not reduce to the form given by Morse and Ingard [1] and introduced in acoustics by Paris [7]. But it is similar to an expression given by Bosquet [8], who further explains that absorption coefficient strongly depends on the weighting function, that is, the statistics one chooses for computing the mean absorption coefficient. In the present case, the obvious choice is the so-called star-type statistics introduced by London [9], that is, taking the mean value of the equation $E_{tz} = \alpha E_{tt}$:

$$\int_0^{\frac{\pi}{2}} E_{tz} \sin \theta d\theta = \alpha^* \int_0^{\frac{\pi}{2}} E_{tt} \sin \theta d\theta \quad (22)$$

One simply obtains:

$$\alpha^* = \frac{2\Re(\beta)}{1 + \frac{2}{3} + |\beta|^2} \quad (23)$$

which is the value of $\alpha(\theta)$ taken for $\theta = 55^\circ$, an angle that has repeatedly appeared since 1953 as the incidence angle that represents the mean absorption [8, 10, 11]. Indeed, $\sin^2 55^\circ = 0.67 \approx \frac{2}{3}$.

Simple calculation shows that the absorption coefficients given by eq. (9) and (23) take values between 0 and a maximum which always is smaller than 1, except at normal incidence ($\theta = 0$) for eq. (9).

5.2 Scattering coefficients

The scattering coefficients given by eq. (10) are independent of the incidence angle θ , except through the reflection coefficient in the expression of $\tan \phi_x$ and $\tan \phi_y$ for curved boundaries. However, the real part of the wall admittance contributes to the scattering coefficients. In other words, absorption on the wall induces scattering. This is a novel result.

Also novel is the observation that scattering coefficients are not bounded: they can take any value between $-\infty$ and $+\infty$, as is easily seen from eq. (10).

Traditionally, scattering is considered as the result of two mechanisms [12]: finite dimensions of reflecting surfaces; and roughness of the reflecting surfaces. There is no doubt from Sect. 4.3 that the second term of eq. (10) corresponds to scattering by surface roughness, since it is driven by the finite distance from the surface to the image source. However, this term subsists for sources located at finite distances from a flat reflecting surface (see Sect. 4.2), as long as its admittance admits an imaginary part. But roughness is usually associated with locally varying imaginary parts of the admittance, which is not necessary here to induce scattering.

We therefore suspect that the mechanism described in Sects. 4 does not correspond to traditional scattering, but to a different effect, because there is no redirection of the acoustical intensity at the boundaries, just a reduction of it by friction on the walls. In other words, total energy and acoustical intensity can be considered as two coupled systems that each follow its own conservation equation with some exchange between the two systems. As is well known from such coupled systems, the less damped system drives the reverberation, as shown in Fig. 1.

As a consequence, the scattering coefficient D is ill-named. A better designation would be *friction* coefficient. But we stick to usage and keep the designation "scattering coefficient".

6 CONCLUSIONS

Starting with the precedent papers of Dujourdy *et al.* [2, 3] that have shown that integrating the stress-energy tensor in a disproportionate enclosure and taking into account the boundary conditions leads to "loss terms" that correspond to absorption and scattering, we expressed the elements of the stress-energy tensor with the help of the wall admittance relation between normal derivative and time derivative of the velocity potential on the wall.

As expected, the absorption coefficients are proportional to the real part of the wall admittance. However, they also depend on the angle of incidence of the wave, leading to an expression of the absorption coefficient eq. (23) which has been signaled in the literature, for example by Bosquet [8], yet is not very usual; and to scattering coefficients additively involving the real and imaginary parts of the wall admittance, as seen in eq. (10), proving the absorptive nature of the scattering coefficient in the stress-energy tensor formalism.

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