Spectral analysis of weakly stationary processes valued in a separable Hilbert space

Amaury Durand, François Roueff

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Abstract

In this paper, we review and clarify the construction of a spectral theory for weakly-stationary processes valued in a separable Hilbert space. We introduce the basic fundamental concepts and results of functional analysis and operator theory needed to follow the way paved by Payen in [52], Mandrekar and Salehi in [45] and Kakihara in [33]. They lead us to view the spectral representation of a weakly stationary Hilbert valued time series as a gramian isometry between its time domain and its spectral domain. Time invariant linear filters with Hilbert-valued inputs and outputs are then defined through their operator transfer functions in the spectral domain. General results on the composition and inversion of such filters follow naturally. Spectral representations have enjoyed a renewed interest in the context of functional time series. The gramian isometry between the time and spectral domains constitutes an interesting and enlightening complement to recent approaches such as the one proposed in [50]. We also provide an overview of recent statistical results for the spectral analysis of functional time-series.

Contents

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1 Introduction

Functional data analysis has become an active field of research in the recent decades due to technological advances which makes it possible to store data at very high frequency (and can be considered as continuous time data i.e. functions) or very complex type of data which could be represented by abstract mathematical structures, typically Hilbert spaces. In this framework, we are considering data belonging in a separable Hilbert space which is often taken as the function space $L^2([0,1])$ of square-integrable functions on $[0,1]$. Naturally, researchers on the topic have been interested in generalizing multivariate data analysis and statistical tools to this framework such as inference, estimation, regression, classification or asymptotic results (see, for example, [53], [22]). As for multivariate data, different tools are used when the data are considered independent or not. In this paper, we are interested in functional data with time dependence (functional stochastic processes), that is we observe a family $(X_t)_{t \in \mathbb{G}}$ of random variables where $\mathbb{G}$ is a set of index (mainly $\mathbb{Z}$ or $\mathbb{R}$) where for each $t \in \mathbb{G}$, X_t is a random variable from a measurable space $(\Omega, \mathcal{F})$ to a separable Hilbert space $\mathcal{H}_0$ (endowed with its Borel σ-field). In the following, we add the assumption (and give a definition) of weak-stationarity. Examples of such processes are functional linear processes like functional AR or, more generally, functional ARMA processes (see [6, 60, 36]). In the univariate and multivariate (finite-dimensional) cases, spectral analysis of weakly-stationary processes has shown many advantages (see *e.g.* [8]). Such an analysis has been recently popularized in [50, 51, 62] for the functional (infinite-dimensional) framework. In particular, the authors define a spectral representation for weakly stationary functional processes based on the spectral density operator. Existence of such a density is shown under strong assumptions on the autocovariance structure of the process (see the discussion in Section 6.3).

The main goals of this paper are twofold : 1) provide a spectral representation for any weakly stationary processes valued in a general (infinite-dimensional) separable Hilbert space, thus relaxing the assumptions of [50, 51, 62]. 2) derive easy to use results on the composition and inversion of shift-invariant linear filters on such processes. The first point is done following earlier works [34, 45, 33] which generalize multivariate approaches [46, 64, 54]. As far as we know, the second point has not been as explicitly studied before.

Let us recall the classical spectral representation of univariate weakly stationary time series, which goes back to [40] (see also [27] for a survey). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L^2(\Omega, \mathcal{F}, \mathbb{P})$ the space of squared integrable $\mathbb{C}$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. This space is a Hilbert space when endowed with the inner product $(X, Y) \mapsto \mathbb{E}\left[X\overline{Y}\right]$, where $\overline{Y}$ is the conjugate of Y. Throughout the paper, we moreover let $(\mathbb{G}, +)$ be a locally compact Abelian (l.c.a.) group, whose null element is denoted by 0 (see Appendix C for details).

Definition 1.1 ((Univariate) weakly stationary process)**.** *We say that $X = (X_t)_{t\in\mathbb{G}}$ is a weakly-stationary process if the following assertions hold.*

(i) For all $t \in \mathbb{G}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is an L^2 process.

(ii) There exists $\mu \in \mathbb{C}$, called the mean *of X, such that for all $t \in \mathbb{G}$, $\mathbb{E}[X_t] = \mu$. We moreover say that X is* centered *if $\mu = 0$.*

(iii) There exists $\gamma_X : \mathbb{G} \to \mathbb{C}$, called the autocovariance function *of X, such that for all $t, h \in \mathbb{G}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \gamma_X(h)$.*

We moreover assume that

(iv) the autocovariance function γ_X is continuous on $\mathbb{G}$.

Without loss of meaningful generality, we will only consider centered processes in the following. Condition (iii) simply says that the covariance of the process is shift invariant ((X_s, X_t) and (X_{s+h}, X_{t+h}) have the same covariance for all $s, t, h \in \mathbb{G}$). The continuity condition (iv) is equivalent to say that X is L^2-continuous, and it always holds when $\mathbb{G} = \mathbb{Z}$. As noted in [40, 27], the analysis of centered, weakly-stationary processes if closely linked to functional analysis and, in particular, to unitary representations.

Definition 1.2 ((Continuous) Unitary representations)**.** *Let $(\mathbb{G}, +)$ be an l.c.a. group and $\mathcal{H}_0$ a Hilbert space. A mapping $U : \begin{array}{ccc} \mathbb{G} & \to & \mathcal{L}_b(\mathcal{H}_0) \\ h & \mapsto & U_h \end{array}$ is said to be a* unitary representation *(u.r.) of $\mathbb{G}$ on $\mathcal{H}_0$ if it satisfies the two following assertions.*

(i) For all $h \in \mathbb{G}$, U_h is a unitary operator from $\mathcal{H}_0$ to $\mathcal{H}_0$.

(ii) The operator U_0 is the identity operator on $\mathcal{H}_0$, that is, $U_0 = \mathrm{Id}_{\mathcal{H}_0}$, and, for all $s, t \in \mathbb{G}$, $U_{s+t} = U_s U_t$.

We say that U is a continuous unitary representation *(c.u.r.) if it moreover satisfies*

(iii) The mapping $h \mapsto U_h$ is continuous on $\mathbb{G}$ for the weak operator topology (w.o.t., that is for all $u, v \in \mathcal{H}_0$, $h \mapsto \langle U_h u, v\rangle_{\mathcal{H}_0}$ is continuous).

Remark 1.1. *Note that a mapping valued in the set of unitary operators is continuous for the w.o.t. if and only if it is continuous for the strong operator topology (s.o.t., that is for all $u \in \mathcal{H}_0$, $h \mapsto U_h u$ is continuous). Hence, a c.u.r. is continuous for the s.o.t. as a consequence of (iii).*

Let $\mathcal{M}(\Omega, \mathcal{F}, \mathbb{P})$ be the sub-Hilbert space of centered variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $X = (X_t)_{t\in\mathbb{G}}$ be a centered L^2 process. Denote by

$$\mathcal{H}^X := \overline{\mathrm{Span}}^{\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})}\,(X_t, t \in \mathbb{G})$$

the sub-Hilbert space generated by $\{X_t, t \in \mathbb{G}\}$, where the notation $\overline{\mathrm{Span}}^{\mathcal{H}}\,(A)$ means the closure in $\mathcal{H}$ of $\mathrm{Span}\,(A)$. Let U_h^X, $h \in \mathbb{G}$, denote the shift operators defined on $\mathcal{H}^X$ by $U_h^X X_t = X_{t+h}$ for all $t \in \mathbb{G}$. The simple remarks made above about Assertions (iii) and (iv) in Definition 1.1 and Definition 1.2 easily yield the following characterization of weak stationarity.

Lemma 1.1. *Let $X = (X_t)_{t\in\mathbb{G}}$ be a centered L^2 process. Then X is weakly stationary if and only if U^X is a c.u.r. of $\mathbb{G}$ on $\mathcal{H}^X$.*

Let $\hat{\mathbb{G}}$ denote the dual group of $\mathbb{G}$ (the continuous characters defined on $\mathbb{G}$, see Appendix C), and denote by $\mathcal{B}(\hat{\mathbb{G}})$ its Borel σ-field. We moreover denote, for all $t \in \mathbb{G}$,

$$\mathrm{e}_t : \begin{array}{ccc} \hat{\mathbb{G}} & \to & \mathbb{C} \\ \chi & \mapsto & \chi(t) \end{array} .$$

Under the above assumptions, both γ_X (as a $\mathbb{C}$-valued function on $\mathbb{G}$) and X (as a $\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})$-valued function on $\mathbb{G}$) admit spectral counterparts, the first one in the form of a finite non-negative regular measure on $(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}))$ and the second one in the form of a countably additive orthogonally scattered (c.a.o.s.) measure on the same space (see Appendix B.2). More precisely, the following theorem holds.

Theorem 1.2 (Spectral measure and spectral representation of a $\mathbb{C}$-valued weakly stationary process)**.** *Let $X = (X_t)_{t\in\mathbb{G}}$ be a centered weakly-stationary process with autocovariance function γ_X. Then there exists a unique finite, non-negative, regular measure ν_X on $(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}))$, called the* spectral measure *of X, such that*

$$\gamma_X(h) = \int \mathrm{e}_h \, \mathrm{d}\nu_X = \int \chi(h)\, \nu_X(\mathrm{d}\chi), \quad h \in \mathbb{G} \; . \tag{1.1}$$

Moreover, there exists a unique $\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})$-valued regular c.a.o.s. measure $\hat{X}$ on $(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}))$ such that for all $t \in \mathbb{G}$,

$$X_t = \int \mathrm{e}_t \, \mathrm{d}\hat{X} = \int \chi(t)\, \hat{X}(\mathrm{d}\chi) \; , \tag{1.2}$$

and the intensity measure of $\hat{X}$ is ν_X, which means that

$$\mathrm{Cov}\left(\hat{X}(A), \hat{X}(B)\right) = \nu_X(A \cap B) \quad \textit{for all } A, B \in \mathcal{B}(\hat{\mathbb{G}}). \tag{1.3}$$

The identity (1.1) is known as Bochner's theorem. The most commonly used index sets are $\mathbb{G} = \mathbb{Z}$ (discrete time) and $\mathbb{G} = \mathbb{R}$ (continuous time). In the discrete time case, $\hat{\mathbb{G}}$ is usually represented in the time series literature by its isomorphic group $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (or $(-\pi,\pi]$ endowed with the addition modulo 2π), so that, for any $\chi \in \hat{\mathbb{Z}}$ and $h \in \mathbb{Z}$, $\chi(h)$ is represented as $\mathrm{e}^{\mathrm{i}h\lambda}$ with $\lambda \in \mathbb{T}$, and $\hat{X}$ and ν_X are a c.a.o.s. and a non-negative measure on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$. For instance, (1.1) then reads

$$\gamma_X(h) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}h\lambda}\, \nu_X(\mathrm{d}\lambda), \quad h \in \mathbb{Z} \; , \tag{1.4}$$

and is then known as Herglotz's theorem (see [10, Theorem 4.3.1].

Note that (1.3) can be rewritten as

$$\mathbb{E}\left[\hat{X}(A)\overline{\hat{X}(B)}\right] = \int \mathbb{1}_A \, \overline{\mathbb{1}_B} \, \mathrm{d}\nu_X \; ,$$

hence as saying that $\mathbb{1}_A \mapsto \hat{X}(A)$ is isometric from $L^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\nu_X)$ to $\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})$. It follows that Relation (1.2) defines the unique isometry I from $L^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\nu_X)$ to $\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})$ which maps $(\chi \mapsto \chi(t))$ to X_t for all $t \in \mathbb{G}$. Moreover, this isometry is unitary from $L^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\nu_X)$ to $\mathcal{H}^X$. The former space is called the *spectral domain* of X and the latter its *time domain* and we conclude that the time and spectral domains are *isometrically isomorphic*. A consequence of the isometric property of I is that, for all $s, t \in \mathbb{G}$,

$$\mathbb{E}\left[X_s \overline{X_t}\right] = \int \chi(s-t)\, \nu_X(\mathrm{d}\chi) \; ,$$

where we used that, for all $\chi \in \hat{\mathbb{G}}$, $\chi(s)\overline{\chi(t)} = \chi(s-t)$, see [57, Eq. (1) and (6)]. This is exactly (1.1) by setting $h = s - t$. We summarize the previous comments as follows.

Remark 1.2. *Theorem 1.2 is presented as containing several successive results and identities. However all these results and identities are several facets of one assertion, namely that*

(i) The spaces $L^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\nu_X)$ and $\mathcal{H}^X$ are isometrically isomorphic by mapping e_t to X_t for all $t \in \mathbb{G}$.

Let us briefly provide the classical steps leading to Theorem 1.2.

Proof of Theorem 1.2 (sketch). As we explained previously, the essential point is to build the unitary mapping between $L^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\nu_X)$ and $\mathcal{H}^X$. To this end, one can rely on the unitary representation provided by the shift operators U_h^X, $h \in \mathbb{G}$, derived in Lemma 1.1.

Then Stone's theorem gives that there exists a regular measure ξ^X on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}))$, valued in the space of orthogonal projections on $\mathcal{H}^X$, such that for all $h \in \mathbb{G}$,

$$U_h^X = \int \chi(h)\, \xi^X(\mathrm{d}\chi)\ . \tag{1.5}$$

The mapping

$$\hat{X} : \begin{array}{ccc} \mathcal{B}(\hat{\mathbb{G}}) & \to & \mathcal{M}(\Omega, \mathcal{F}, \mathbb{P}) \\ A & \mapsto & \xi^X(A)X_0 \end{array}$$

is then a regular c.a.o.s. measure on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{M}(\Omega, \mathcal{F}, \mathbb{P}))$ and from (1.5) we get

$$X_t = U_t^X X_0 = \int \chi(t)\, \xi^X(\mathrm{d}\chi) X_0 = \int \chi(t)\, \hat{X}(\mathrm{d}\chi), \quad t \in \mathbb{G}\ ,$$

which is exactly (1.2). Then, by properties of c.a.o.s. measures, this relation defines an isometry and (1.1) follows from this result taking for ν_X the intensity measure of $\hat{X}$. ☐

It is also common to find a proof of Theorem 1.2 where (1.1) is proved first and is used to prove (1.2) (see *e.g.* [8]). This is a consequence of the close relationship between the functional analysis tools used in the proofs and will be discussed in Section 6.1.

Having recalled the univariate case, we can now give more details about the goals of this paper. We consider the functional case where the process X is a sequence $(X_t)_{t\in\mathbb{G}}$ of variables in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, that is, for all $t \in \mathbb{G}$, X_t is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, valued in the separable Hilbert space $\mathcal{H}_0$ and satisfies $\mathbb{E}\left[\|X_t\|_{\mathcal{H}_0}^2\right] < +\infty$. Recall that the expectation of $Y \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique vector $\mathbb{E}[Y] \in \mathcal{H}_0$ satisfying

$$\langle \mathbb{E}[Y], x\rangle_{\mathcal{H}_0} = \mathbb{E}\left[\langle Y, x\rangle_{\mathcal{H}_0}\right], \quad \text{for all } x \in \mathcal{H}_0$$

and that the covariance operator between $Y, Z \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique linear operator $\mathrm{Cov}\,(Y, Z) \in \mathcal{L}_b(\mathcal{H}_0)$, satisfying

$$\langle \mathrm{Cov}\,(Y, Z)\, y, x\rangle_{\mathcal{H}_0} = \mathrm{Cov}\left(\langle Y, x\rangle_{\mathcal{H}_0}, \langle Z, y\rangle_{\mathcal{H}_0}\right), \quad \text{for all } x, y \in \mathcal{H}_0\ ,$$

or, more concisely, $\mathrm{Cov}\,(Y, Z) = \mathbb{E}[Y \otimes Z] - \mathbb{E}[Y] \otimes \mathbb{E}[Z]$ where, for all $x, y \in \mathcal{H}_0$, $x \otimes y$ denotes the operator of $\mathcal{L}_b(\mathcal{H}_0)$ which satisfies for all $z \in \mathcal{H}_0$, $(x \otimes y)z = \langle z, y\rangle_{\mathcal{H}_0} x$. In this setting, Definition 1.1 is extended as follow.

Definition 1.3 (Hilbert valued weakly stationary process)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ a separable Hilbert space and $(\mathbb{G}, +)$ an l.c.a. group. Then a process $X := (X_t)_{t\in\mathbb{G}}$ is said to be an $\mathcal{H}_0$-valued weakly-stationary process if*

(i) *For all $t \in \mathbb{G}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$.*

(ii) *For all $t \in \mathbb{G}$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$. We say that X is centered if $\mathbb{E}[X_0] = 0$.*

(iii) *For all $t, h \in \mathbb{G}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \mathrm{Cov}\,(X_h, X_0)$.*

(iv) *The* autocovariance operator function $\Gamma_X : h \mapsto \mathrm{Cov}\,(X_h, X_0)$ *is weakly continuous i.e. for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, $h \mapsto \mathrm{Tr}(\mathrm{P}\Gamma_X(h))$ is continuous.*

Given a separable Hilbert space $\mathcal{H}_0$ and a centered weakly stationary $\mathcal{H}_0$-valued process $X := (X_t)_{t\in\mathbb{G}}$, we want to derive

R1 A spectral version of the covariance structure of X similar to (1.1) :

$$\mathrm{Cov}\,(X_s, X_t) = \int \chi(s-t)\, \nu_X(\mathrm{d}\chi), \quad s, t \in \mathbb{G}\ , \tag{1.6}$$

where ν_X will be called the *spectral operator measure* of X.

R2 A spectral representation of X similar to (1.2) :

$$X_t = \int \chi(t)\, \hat{X}(\mathrm{d}\chi), \quad \mathbb{P}\text{-a.e.} \quad t \in \mathbb{G}\ , \tag{1.7}$$

as well as a description of the isomorphic relationship that this mapping induces.

R3 A practical definition of shift-invariant linear filters, with results for composition and inversion in the spectral domain.

In [27], the univariate and functional cases are described in a unified setting, by directly considering $(X_t)_{t\in\mathbb{Z}}$ as a $\mathcal{H}$-valued sequence, where $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$ in the univariate case and $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ in the functional case. However, in the second case, as explained later, $\mathcal{H}$ should be seen as a *normal Hilbert module* rather than just a Hilbert space and this fact has consequences on the previous points, as suggested in the following remarks.

Remark 1.3. *<u>About **R1**.</u> Since the left hand side term of* (1.6) *is an operator on $\mathcal{H}_0$ and for all $\chi \in \hat{\mathbb{G}}$ and $h \in \mathbb{G}$, $\chi(h) \in \mathbb{C}$, the measure ν_X must be operator-valued. Since in the univariate case ν_X is a non-negative measure, we expect it to verify an analogous property for the Hilbert valued case that is to be a* Positive Operator Valued Measure *(p.o.v.m.).*
*<u>About **R2**.</u> In the univariate case, $\hat{X}$ is a measure valued in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ that makes the time domain and the spectral domain isometrically isomorphic, as summarized in Remark 1.2. In the Hilbert valued case, we naturally expect $\hat{X}$ to be a measure valued in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. The fact that this space is not only a Hilbert space but also a normal Hilbert module will have the consequence that $\hat{X}$ will make the time domain and the spectral domain* gramian-isometrically isomorphic. *It will also yield a more involved description of the spectral domain.*
*<u>About **R3**.</u> Consider, in the univariate case, the output $Y = (Y_t)_{t\in\mathbb{G}}$ of a linear shift-invariant filter with input $X = (X_t)_{t\in\mathbb{G}}$. Then, by linearity, Y_0 belongs to the time domain $\mathcal{H}^X$, and, by shift-invariance, we have $Y_t = U_t^X Y_0$ for all $t \in \mathbb{G}$. Consequently, in the spectral domain, the linear shift-invariant filtering of X consists in multiplying $\hat{X}$ by the $\mathbb{C}$-valued function $\varphi \in L^2(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \nu_X)$ which represents Y_0 in the spectral domain. Namely, $Y_t = \int \chi(t)\varphi(\chi)\,\hat{X}(\mathrm{d}\chi)$ for all $t \in \mathbb{G}$, or, equivalently, $\hat{Y}(\mathrm{d}\chi) = \varphi(\chi)\hat{X}(\mathrm{d}\chi)$. The function φ is called the* transfer function *of the filter and the condition*

$$\varphi \in L^2(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \nu_X) \tag{1.8}$$

should be seen as a condition on the input X (through its spectral measure ν_X) in order to have a well defined weakly stationary output Y through the filter with given transfer function φ. In the Hilbert valued case, where the input is $\mathcal{H}_0$-valued and the output $\mathcal{G}_0$-valued, the shift-invariant filtering of X in the spectral domain becomes $\hat{Y}(\mathrm{d}\chi) = \Phi(\chi)\hat{X}(\mathrm{d}\chi)$, where, for all $\chi \in \hat{T}$, $\Phi(\chi)$ is an operator from $\mathcal{H}_0$ to $\mathcal{G}_0$, and we need to investigate two important questions:

a) In which space should the transfer operator *function Φ be defined and what condition on X (again through ν_X) should replace (1.8) ?*

b) How are composition and inversion of filters described in the spectral domain? (That is, through transfer operator functions.)

Remark 1.4. *In Condition (iv) of Definition 1.3, one could have chosen a weaker notion of continuity for the autocovariance operator function, such as continuity of Γ_X for the w.o.t. The usefulness of the weak-continuity as assumed in (iv) to get **R1**, **R2**, **R3** will be made clearer in Section 3. However, in Section 6.2, we will see that, for autocovariance operator functions, weak-continuity is actually equivalent to continuity for the w.o.t.*

The paper is organized as follows. In Section 2 we gather definitions and results of operator theory needed all along the paper. This effort pays off in Section 3 as we easily derive precise statements on the spectral representation for Hilbert valued weakly stationary processes, with clear and simple answers to the questions raised in Remark 1.3 about **R1**, **R2** and **R3**. A brief overview of the recent approaches for deriving statistical results on the spectral analysis of functional time series can be found in Section 4. We provide important proofs of the results stated in Section 2 and Section 3 in Section 5. We conclude the main part of this contribution by Section 6, where we discuss alternative approaches, in particular comparisons with recent approaches. Additional useful results on functional analysis and l.c.a. groups are gathered in the appendices.

2 Operator theory prerequisites

2.1 Operator spaces, measurability and L^p spaces

Here we introduce classical definitions for operators on Hilbert spaces (see e.g. [24] for details) and integrals of functions with respect to a measure in the case where the function or the measure is vector-valued (see e.g. [19, Chapter 1] for a nice overview and [18], [17] for a thorough study). This section also contains most of the notation used throughout the paper.

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. The inner product and norm, e.g. associated to $\mathcal{H}_0$, are denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ and $\|\cdot\|_{\mathcal{H}_0}$. Let $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$ denote the set of linear operators P from $\mathcal{H}_0$ to $\mathcal{G}_0$ whose domain, denoted by $\mathcal{D}(\mathrm{P})$, is a linear subspace of $\mathcal{H}_0$, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ the set of all $\mathcal{H}_0 \to \mathcal{G}_0$ continuous operators. We also denote by $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ the set of all compact operators in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and for all $p \in [1, \infty)$, $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ the Schatten-p class. The space $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and the Schatten-p classes are Banach spaces when respectively endowed with the norms

$$\|\mathrm{P}\| := \sup_{\|x\|_{\mathcal{H}_0} \leq 1} \|\mathrm{P}x\|_{\mathcal{G}_0} \quad \text{and} \quad \|\mathrm{P}\|_p := \left(\sum_{\sigma \in \mathrm{sing}(\mathrm{P})} \sigma^p \right)^{1/p}$$

where $\mathrm{sing}(\mathrm{P})$ is the set of singular values of P. Following these definitions, we have, for all $1 \leq p \leq p'$

$$\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{S}_{p'}(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0). \tag{2.1}$$

The space $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ is endowed with the operator norm and the first three inclusions in (2.1) are continuous embeddings. If $\mathcal{G}_0 = \mathcal{H}_0$, we omit the $\mathcal{G}_0$ in the notations above. As a Banach space, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ can be endowed with its norm topology but other common topologies are useful. The two most common ones are the strong and weak topologies (respectively denoted by s.o.t. and w.o.t.). We say that a sequence $(\mathrm{P}_n)_{n \in \mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)^{\mathbb{N}}$ converges to an operator $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ for the s.o.t. if for all $x \in \mathcal{H}_0$, $\lim_{n \to +\infty} \mathrm{P}_n x = \mathrm{P}x$ in $\mathcal{G}_0$ and for the w.o.t. if for all $x \in \mathcal{H}_0$, for all $y \in \mathcal{G}_0$, $\lim_{n \to +\infty} \langle \mathrm{P}_n x, y \rangle_{\mathcal{G}_0} = \langle \mathrm{P}x, y \rangle_{\mathcal{G}_0}$.

An operator $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, is said to be *positive* if for all $x \in \mathcal{H}_0$, $\langle \mathrm{P}x, x \rangle_{\mathcal{H}_0} \geq 0$ and we will use the notations $\mathcal{L}_b^+(\mathcal{H}_0)$, $\mathcal{K}^+(\mathcal{H}_0)$, $\mathcal{S}_p^+(\mathcal{H}_0)$ for positive, positive compact and positive Schatten-p operators. If $\mathrm{P} \in \mathcal{L}_b^+(\mathcal{H}_0)$ then there exists a unique operator of $\mathcal{L}_b^+(\mathcal{H}_0)$, denoted by $\mathrm{P}^{1/2}$, which satisfies $\mathrm{P} = \left(\mathrm{P}^{1/2}\right)^2$. If, in addition, P is compact, then so is $\mathrm{P}^{1/2}$. For any $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ we denote its adjoint by P^{H} (which is compact if P is compact). An operator of $\mathcal{L}_b(\mathcal{H}_0)$ is said to be auto-adjoint is it is equal to its adjoint and it is known that any positive operator is auto-adjoint. If $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, then $\mathrm{P}^{\mathsf{H}}\mathrm{P} \in \mathcal{L}_b^+(\mathcal{H}_0)$ and $\mathrm{P}\mathrm{P}^{\mathsf{H}} \in \mathcal{L}_b^+(\mathcal{G}_0)$ (which are compact if P is compact). We define the *absolute value* of P as the operator $|\mathrm{P}| := \left(\mathrm{P}^{\mathsf{H}}\mathrm{P}\right)^{1/2} \in \mathcal{L}_b^+(\mathcal{H}_0)$. Moreover, if $\mathrm{P} \in \mathcal{S}_1(\mathcal{H}_0)$, $\mathrm{Tr}(\mathrm{P})$ will denote its trace, if $\mathrm{P} \in \mathcal{S}_1^+(\mathcal{H}_0)$, it is known that $\mathrm{Tr}(\mathrm{P}) = \|\mathrm{P}\|_1$. Schatten-1 and Schatten-2 operators are usually referred to as *trace-class* and *Hilbert-Schmidt* operators respectively.

For functions defined on a measurable space $(\Lambda, \mathcal{A})$ and valued in a Banach space $(E, \|\cdot\|_E)$, measurability is defined as follows. A function $f : \Lambda \mapsto E$ is said to be *measurable* if it is the pointwise limit of a sequence of E-valued *simple functions*, *i.e.* functions belonging in the space $\mathrm{Span}\,(\mathbb{1}_A x \,:\, A \in \mathcal{A},\, x \in E)$. When E is separable, this notion is equivalent to the usual Borel-measurability, *i.e.* to having $f^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{B}(E)$, the Borel σ-field on E. We denote by $\mathbb{F}(\Lambda, \mathcal{A}, E)$ (resp. $\mathbb{F}_b(\Lambda, \mathcal{A}, E)$) the space of measurable (resp. bounded measurable) functions from Λ to E. For a non-negative measure μ on $(\Lambda, \mathcal{A})$ and $p \in [1, \infty]$, we denote by $\mathcal{L}^p(\Lambda, \mathcal{A}, E, \mu)$ the space of functions $f \in \mathbb{F}(\Lambda, \mathcal{A}, E)$ such that $\int \|f\|_E^p \,\mathrm{d}\mu$ (or μ-$\mathrm{essup}\, \|f\|_E$ for $p = \infty$) is finite and by $L^p(\Lambda, \mathcal{A}, E, \mu)$ its quotient space with respect to μ-a.e. equality, or, equivalently, with respect to the subspace of functions f such that $f = 0$ μ-a.e., which we write

$$L^p(\Lambda, \mathcal{A}, E, \mu) = \mathcal{L}^p(\Lambda, \mathcal{A}, E, \mu) \Big/ \{f \,:\, f = 0 \ \mu\text{-a.e.}\} \cdot$$

The corresponding norms are denoted by $\|f\|_{L^p(\Lambda, \mathcal{A}, E, \mu)}$. For $p \in [1, \infty)$, the space of simple measurable functions with finite-measure support, *i.e.* $\mathrm{Span}\,(\mathbb{1}_A x \,:\, A \in \mathcal{A},\, \mu(A) < \infty,\, x \in E)$, is dense in $L^p(\Lambda, \mathcal{A}, E, \mu)$. For $f \in$

$\mathrm{Span}\,(\mathbb{1}_A x \,:\, A \in \mathcal{A},\, \mu(A) < \infty,\, x \in E)$ with range $\{\alpha_1, \cdots, \alpha_n\}$, the integral (often referred to as the *Bochner integral*) of the E-valued function f with respect to μ is defined by

$$\int f \,\mathrm{d}\mu = \sum_{k=1}^{n} \alpha_k \, \mu\left(f^{-1}(\{\alpha_k\})\right) \in E \,. \tag{2.2}$$

This integral is extended to $L^1(\Lambda, \mathcal{A}, E, \mu)$ by continuity (and thus also to L^p if μ is finite).

An E-valued measure is a mapping $\mu : \mathcal{A} \to E$ such that for any sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ of pairwise disjoint sets then $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$ where the series converges in E, that is

$$\lim_{N\to+\infty} \left\| \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) - \sum_{n=0}^{N} \mu(A_n) \right\|_E = 0 \,.$$

We denote by $\mathbb{M}(\Lambda, \mathcal{A}, E)$ the set of E-valued measures. For such a measure μ, the mapping

$$\|\mu\|_E : A \mapsto \sup\left\{ \sum_{i\in\mathbb{N}} \|\mu(A_i)\|_E \,:\, (A_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} \text{ is a countable partition of } A \right\}$$

defines a non-negative measure on $(\Lambda, \mathcal{A})$ called the *variation measure* of μ. The notation $\|\mu\|_E$ will be adapted to the notation chosen for the norm in E (for example if μ is a complex measure we will use $|\mu|$ and if μ is valued in a Schatten-p space we will use $\|\mu\|_p$). The variation of a complex-valued measure is always finite and the variation of a non-negative measure is itself. We will denote by $\mathbb{M}_b(\Lambda, \mathcal{A}, E)$ the set of E-valued measures with finite variation. It is a Banach space when endowed with the norm $\|\mu\|_{TV,E} = \|\mu\|_E(\Lambda)$. If $\mu \in \mathbb{M}_b(\Lambda, \mathcal{A}, E)$, then for a simple function $f : \Lambda \mapsto \mathbb{C}$ with range $\{\alpha_1, \ldots, \alpha_n\}$, the integral of f with respect to μ is defined by the same formula as in (2.2) (but this time the α_k's are scalars and the μ's are E-valued). This definition is extended to $L^1(\Lambda, \mathcal{A}, \|\mu\|_E)$ by continuity.

When Λ is a locally-compact topological space, a vector measure $\mu \in \mathbb{M}(\Lambda, \mathcal{A}, E)$ is said to be *regular* if for all $A \in \mathcal{A}$, for all $\epsilon > 0$, there exist a compact set $K \in \mathcal{A}$ and an open set $U \in \mathcal{A}$ with $K \subset A \subset U$ such that for all $B \in \mathcal{A}$ satisfying $B \subset U \setminus K$, $\|\mu(B)\|_E \leq \epsilon$. We denote by $\mathbb{M}_r(\Lambda, \mathcal{A}, E)$ the linear space of such measures. The notion of regularity is extended to non-finite, non-negative measures by restricting A to be such that $\mu(A) < +\infty$. From the straightforward inequality $\|\mu(A)\|_E \leq \|\mu\|_E(A)$ for all $A \in \mathcal{A}$, we get that if $\mu \in \mathbb{M}_b(\Lambda, \mathcal{A}, E)$ has a regular variation, then μ is regular. The converse is not always true but holds for complex measures. An interesting result (see [33, Remark 3.6.2]) is that an E-valued measure ν is regular if and only if for all $\phi \in E^*$, $\phi \circ \nu$ is a regular complex measure.

Finally, we recall another notion of measurability for functions valued in the operator spaces $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ or $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$. Namely, a function $\Phi : \Lambda \to \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ is said to be *simply measurable* if for all $x \in \mathcal{H}_0$, $\lambda \mapsto \Phi(\lambda)x$ is measurable as a $\mathcal{G}_0$-valued function. The set of such functions is denoted by $\mathbb{F}_s\left(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0\right)$. For a function $\Phi : \Lambda \to \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$, adapting [45], [33, Section 3.4], we will say that Φ is $\mathcal{O}$-measurable if it satisfies the two following conditions.

(i) For all $x \in \mathcal{H}_0$, $\{\lambda \in \Lambda \,:\, x \in \mathcal{D}(\Phi(\lambda))\} \in \mathcal{A}$.

(ii) There exist a sequence $(\Phi_n)_{n\in\mathbb{N}}$ valued in $\mathbb{F}_s\left(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0\right)$ such that for all $\lambda \in \Lambda$ and $x \in \mathcal{D}(\Phi(\lambda))$, $\Phi_n(\lambda)x$ converges to $\Phi(\lambda)x$ in $\mathcal{G}_0$ as $n \to \infty$.

We denote by $\mathbb{F}_{\mathcal{O}}\left(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0\right)$ the space of such functions Φ. Note that for all Banach space $\mathcal{E}$ which is continuously embedded in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ (*e.g.* $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ for $p \geq 1$ or $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$), the following inclusions hold

$$\mathbb{F}(\Lambda, \mathcal{A}, \mathcal{E}) \subset \mathbb{F}_s\left(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0\right) \subset \mathbb{F}_{\mathcal{O}}\left(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0\right) \,. \tag{2.3}$$

In this paper we will mainly take $\mathcal{E}$ as the set of trace-class, Hilbert-Schmidt or compact $\mathcal{H}_0 \to \mathcal{G}_0$ operators for which measurability and simple measurability are equivalent as stated in the following lemma.

Lemma 2.1. *Let $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ or $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ where $p \in \{1, 2\}$ and $\mathcal{H}_0, \mathcal{G}_0$ are separable Hilbert spaces. Then a function $\Phi : \Lambda \to \mathcal{E}$ is measurable if and only if it is simply measurable.*

We also need to consider operator-valued measures for our study, and more particularly p.o.v.m.'s which are studied in the next section.

2.2 Positive Operator Valued Measures

The notion of Positive Operator Valued Measures is widely used in Quantum Mechanics and a good study of such measures can be found in [4]. Here we provide useful definitions and results for our purpose.

Definition 2.1 (Positive Operator Valued Measures)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space. A Positive Operator Valued Measure (p.o.v.m.) on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is a mapping $\nu : \mathcal{A} \to \mathcal{L}_b^+(\mathcal{H}_0)$ such that for all sequences of disjoint sets $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$,*

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n) \tag{2.4}$$

where the series converges in $\mathcal{L}_b^+(\mathcal{H}_0)$ for the s.o.t.

Due to properties of positive operators, convergence in the w.o.t. would be sufficient in Definition 2.1, see [4, Proposition 1]. Note that, with this definition, a p.o.v.m. is not a vector-valued measure in the sense of Section 2.1 since we do not suppose that the series in (2.4) converges in operator norm. However, this definition is sufficient to derive a useful characterization which links a p.o.v.m. to a sesquilinear, hermitian, positive semi-definite, continuous mapping valued in $\mathbb{M}(\Lambda, \mathcal{A})$.

Definition 2.2. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space. A mapping $\phi : \mathcal{H}_0^2 \to \mathbb{M}(\Lambda, \mathcal{A})$ is said to be sesquilinear, hermitian, positive semi-definite, continuous if for all $A \in \mathcal{A}$, the mapping $(x, y) \mapsto \phi(x, y)(A)$ is sesquilinear, hermitian, positive semi-definite, continuous.*

The characterization of p.o.v.m.'s then reads as follows (see [4, Theorem 2]).

Proposition 2.2. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space, then the following assertions hold.*

(i) For all p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and all $x, y \in \mathcal{H}_0$, the mapping $y^{\mathsf{H}}\nu x : A \mapsto \langle \nu(A)x, y\rangle_{\mathcal{H}_0}$ is a complex-valued measure on $(\Lambda, \mathcal{A})$. Moreover, the mapping $(x, y) \mapsto y^{\mathsf{H}}\nu x$ is sesquilinear, hermitian, positive semi-definite, continuous.

(ii) Conversely, if $\phi : \mathcal{H}_0^2 \to \mathbb{M}(\Lambda, \mathcal{A})$ is a sesquilinear, hermitian, positive semi-definite bounded mapping, then there exists a unique p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ such that for all $x, y \in \mathcal{H}_0$, $\phi(x, y) = y^{\mathsf{H}}\nu x$.

This characterization can be used to construct integrals of bounded complex-valued functions with respect to p.o.v.m.'s and we refer to [4, Section 5] for details. When Λ is a locally-compact topological space, this also gives a simple notion of regularity for p.o.v.m.'s, namely a p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is said to be *regular* if for all $x, y \in \mathcal{H}_0$, the measure $y^{\mathsf{H}}\nu x$ is a regular complex measure. We will say that a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is *trace-class* if it is $\mathcal{S}_1(\mathcal{H}_0)$-valued. The following lemma states that trace-class p.o.v.m.'s can be seen as vector-valued measures.

Lemma 2.3. *A p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is trace-class if and only if $\nu(\Lambda) \in \mathcal{S}_1(\mathcal{H}_0)$. In this case, ν is a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure (in the sense that (2.4) holds in $\|\cdot\|_1$-norm) with finite variation measure $\|\nu\|_1 : A \mapsto \|\nu(A)\|_1$. Moreover, regularity of ν as a p.o.v.m. is equivalent to regularity of ν as a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which is itself equivalent to regularity of $\|\nu\|_1$.*

Thanks to this result, integration of complex-valued functions with respect to a trace-class p.o.v.m. is possible using the theory of vector-valued measures with finite variation recalled in Section 2.1. Finally, the following Radon-Nikodym property holds.

Theorem 2.4. *Let $(\Lambda, \mathcal{A})$ be a measure space, $\mathcal{H}_0$ a separable Hilbert space and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let μ be a finite non-negative measure on $(\Lambda, \mathcal{A})$. Then $\|\nu\|_1 \ll \mu$ (i.e. for all $A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow \|\nu\|_1(A) = 0$), if and only if there exists $g \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ such that $\mathrm{d}\nu = g\,\mathrm{d}\mu$, i.e. for all $A \in \mathcal{A}$,*

$$\nu(A) = \int_A g\,\mathrm{d}\mu\,. \tag{2.5}$$

In this case, g is unique and is called the density of ν with respect to μ and denoted as $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Moreover, the following assertions hold.

(a) *For μ-almost every $\lambda \in \Lambda$, $g(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$.*

(b) *The density of $\|\nu\|_1$ with respect to μ is $\|g\|_1$. In particular, $g = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1} \|g\|_1$.*

(c) *If $\|\nu\|_1 \leq \mu$, then $\|g\|_1 \leq 1$ μ-a.e., and if $\mu = \|\nu\|_1$, then $\|g\|_1 = 1$ μ-a.e.*

2.3 Normal Hilbert modules

Modules extend the notion of vector spaces to the case where scalar multiplication is replaced by a multiplicative operation with elements of a ring. When the ring is a C^*-algebra, it is possible to endow a module with a structure similar to a Hilbert space (see [35]). Following [33], hereafter, we consider the case where the C^*-algebra is the space $\mathcal{L}_b(\mathcal{H}_0)$ for a separable Hilbert space $\mathcal{H}_0$.

Definition 2.3 ($\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. A $\mathcal{L}_b(\mathcal{H}_0)$-module is a commutative group $(\mathcal{H}, +)$ such that there exists a multiplicative operation (called the* module action*)*

$$\begin{array}{ccc} \mathcal{L}_b(\mathcal{H}_0) \times \mathcal{H} & \to & \mathcal{H} \\ (\mathrm{P}, x) & \mapsto & \mathrm{P} \bullet x \end{array}$$

which satisfies the usual distributive properties : for all $\mathrm{P}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$, and $x, y \in \mathcal{H}$,

$$\begin{aligned} \mathrm{P} \bullet (x + y) &= \mathrm{P} \bullet x + \mathrm{P} \bullet y, \\ (\mathrm{P} + \mathrm{Q}) \bullet x &= \mathrm{P} \bullet x + \mathrm{Q} \bullet x, \\ (\mathrm{PQ}) \bullet x &= \mathrm{P} \bullet (\mathrm{Q} \bullet x), \\ \mathrm{Id}_{\mathcal{H}_0} \bullet x &= x. \end{aligned}$$

Next, we endow a $\mathcal{L}_b(\mathcal{H}_0)$-module with an $\mathcal{L}_b(\mathcal{H}_0)$-valued product.

Definition 2.4 ((Normal) pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. A pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module $\mathcal{H}$ is a $\mathcal{L}_b(\mathcal{H}_0)$-module endowed with a mapping $[\cdot, \cdot]_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{L}_b(\mathcal{H}_0)$ satisfying for all $x, y, z \in \mathcal{H}$, and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$,*

(i) $[x, x]_{\mathcal{H}} \in \mathcal{L}_b^+(\mathcal{H}_0)$*,*

(ii) $[x, x]_{\mathcal{H}} = 0$ *if and only if* $x = 0$*,*

(iii) $[x + \mathrm{P} \bullet y, z]_{\mathcal{H}} = [x, z]_{\mathcal{H}} + \mathrm{P}[y, z]_{\mathcal{H}}$*,*

(iv) $[y, x]_{\mathcal{H}} = [x, y]_{\mathcal{H}}^{\mathsf{H}}$*.*

If moreover, for all $x, y \in \mathcal{H}$, $[x, y]_{\mathcal{H}} \in \mathcal{S}_1(\mathcal{H}_0)$, we say that $[\cdot, \cdot]_{\mathcal{H}}$ is a gramian *and that $\mathcal{H}$ is a* normal *pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module.*

The mapping $[\cdot, \cdot]_{\mathcal{H}}$ generalizes the notion of scalar products for $\mathcal{L}_b(\mathcal{H}_0)$-modules and is often called a $\mathcal{L}_b(\mathcal{H}_0)$-valued scalar product. In the following, we only consider normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules even if some notions can be defined when $[\cdot, \cdot]_{\mathcal{H}}$ is not a gramian. Note that a $\mathcal{L}_b(\mathcal{H}_0)$-module is a vector space if we define the scalar-vector multiplication by $\alpha x = (\alpha \mathrm{Id}_{\mathcal{H}_0}) \bullet x$ for all $\alpha \in \mathbb{C}$, $x \in \mathcal{H}$ and that, in the particular case where $[\cdot, \cdot]$ is a gramian, then $\langle \cdot, \cdot \rangle := \mathrm{Tr}[\cdot, \cdot]$ is a scalar product. Hence a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is also a pre-Hilbert space. If it is complete (for the norm $\|x\|_{\mathcal{H}} = \left\|[x, x]_{\mathcal{H}}\right\|_1^{1/2}$), then it is called a *normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module*. For normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules, the notions of sub-modules and $\mathcal{L}_b(\mathcal{H}_0)$-linear span as well as $\mathcal{L}_b(\mathcal{H}_0)$-linear operators, gramian-isometries, gramian-unitary operators, gramian-orthogonality, gramian-projections come as natural extensions of their vector space counterparts. For completeness, we provide here the necessary definitions and refer to chapter II of [33] for a complete study.

Definition 2.5 (Submodules and $\mathcal{L}_b(\mathcal{H}_0)$-linear operators)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space and $\mathcal{H}, \mathcal{G}$ two $\mathcal{L}_b(\mathcal{H}_0)$-modules. Then a subset of $\mathcal{H}$ is called a* submodule *if it is a $\mathcal{L}_b(\mathcal{H}_0)$-module. An operator $\mathrm{F} \in \mathcal{L}_b(\mathcal{H}, \mathcal{G})$ is said to be* $\mathcal{L}_b(\mathcal{H}_0)$-linear *if for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$ and $x \in \mathcal{H}$, $\mathrm{F}(\mathrm{P} \bullet x) = \mathrm{P} \bullet (\mathrm{F}x)$.*

Definition 2.6 (Gramian-isometries)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}, \mathcal{G}$ be two pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules and $U : \mathcal{H} \to \mathcal{G}$ a $\mathcal{L}_b(\mathcal{H}_0)$-linear operator. Then U is said to be*

(i) *a* gramian-isometry *(or gramian-isometric) if for all $x, y \in \mathcal{H}$, $[Ux, Uy]_{\mathcal{G}} = [x, y]_{\mathcal{H}}$,*

(ii) gramian-unitary *if it is a bijective gramian-isometry.*

The space $\mathcal{H}$ is said to be gramian-isometrically-embedded *in $\mathcal{G}$ (denoted $\mathcal{H} \underset{\approx}{\subsetneq} \mathcal{G}$) if there exists a gramian-isometry from $\mathcal{H}$ to $\mathcal{G}$. The spaces $\mathcal{H}$ and $\mathcal{G}$ are said to be* gramian-isomorphic *(denoted $\mathcal{H} \cong \mathcal{G}$) if there exists a gramian-unitary operator from $\mathcal{H}$ to $\mathcal{G}$.*

Definition 2.7 ((Continuous) gramian unitary representations)**.** *Let $(\mathbb{G}, +)$ be an l.c.a. group, $\mathcal{H}_0$ a Hilbert space and $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module with gramian $[\cdot,\cdot]_{\mathcal{H}}$. A mapping $U : \begin{array}{ccc} \mathbb{G} & \mapsto & \mathcal{L}_b(\mathcal{H}) \\ h & \mapsto & U_h \end{array}$ is said to be a* gramian unitary representation *(g.u.r.) of $\mathbb{G}$ on $\mathcal{H}$ if it is an u.r. of $\mathbb{G}$ on $\mathcal{H}$ such that for all $h \in \mathbb{G}$, U_h is gramian-unitary. A g.u.r. is continuous, then called a c.g.u.r., if it is continuous as an u.r.*

For later reference we state a simple extension result for gramian isometric operators.

Proposition 2.5 (Gramian-isometric extension)**.** *Let $\mathcal{H}$ be a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, $\mathcal{G}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module. Let $(v_j)_{j\in J}$ and $(w_j)_{j\in J}$ be two sets of vectors in $\mathcal{H}$ and $\mathcal{G}$ respectively with J an arbitrary index set. If for all $i, j \in J$, $[v_i, v_j]_{\mathcal{H}} = [w_i, w_j]_{\mathcal{G}}$ then there exists a unique gramian-isometry*

$$S : \overline{\mathrm{Span}}^{\mathcal{H}} \left(\mathrm{P} \bullet v_j, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right) \to \mathcal{G}$$

such that for all $j \in J$, $Sv_j = w_j$. If moreover $\mathcal{H}$ is complete then

$$S\left(\overline{\mathrm{Span}}^{\mathcal{H}} \left(\mathrm{P} \bullet v_j, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right)\right) = \overline{\mathrm{Span}}^{\mathcal{G}} \left(\mathrm{P} \bullet w_j, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right)$$

We can now state an important result, which generalizes Stone's theorem to c.g.u.r.'s. We refer to [33, Proposition 2.5.4] for a proof and Appendix B.1 for the definition of gramian-projection valued measures.

Theorem 2.6 (Stone's theorem for modules)**.** *Let $(\mathbb{G}, +)$ be an l.c.a. group, $\mathcal{H}_0$ a Hilbert space, $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module with gramian $[\cdot,\cdot]_{\mathcal{H}}$ and $U : \begin{array}{ccc} \mathbb{G} & \mapsto & \mathcal{L}_b(\mathcal{H}) \\ h & \mapsto & U_h \end{array}$ a c.g.u.r of $\mathbb{G}$ on $\mathcal{H}$. Then there exists a unique regular gramian-projection valued measure ξ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}))$ such that*

$$U_h = \int \chi(h)\, \xi(\mathrm{d}\chi), \quad h \in \mathbb{G} . \tag{2.6}$$

We conclude this section with some examples of normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules.

Example 2.1. *Let us provide some examples of Hilbert modules built from two separable Hilbert spaces $\mathcal{H}_0$ and $,\mathcal{G}_0$.*

(1) Identifying $\mathcal{L}_b(\mathbb{C})$ with $\mathbb{C}$, $\mathcal{H}_0$ is a normal Hilbert $\mathcal{L}_b(\mathbb{C})$-module with module action $a \bullet x = a\, x$ and its gramian reduces to the scalar product of $\mathcal{H}_0$.

(2) $\mathcal{H}_0$ is also a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module with module action $\mathrm{P} \bullet x = \mathrm{P}x$ and gramian $[x, y]_{\mathcal{H}_0} = x \otimes y$ where $(x \otimes y)u = \langle u, y\rangle_{\mathcal{H}_0} x$ for all $u \in \mathcal{H}_0$.

(3) $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module with module action $\mathrm{P} \bullet \mathrm{Q} = \mathrm{PQ}$ and gramian $[\mathrm{P}, \mathrm{Q}]_{\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0)} = \mathrm{PQ}^{\mathsf{H}}$.

(4) Let $(\Lambda, \mathcal{A})$ be a measurable space and μ a finite non-negative measure on $(\Lambda, \mathcal{A})$. Then for all normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module $\mathcal{H}$, the space $L^2(\Lambda, \mathcal{A}, \mathcal{H}, \mu)$ is a normal $\mathcal{L}_b(\mathcal{H}_0)$-Hilbert module for the module action $(\mathrm{P}\bullet f)(\cdot) = \mathrm{P}\bullet[f(\cdot)]$ and gramian $[f, g]_{L^2(\Lambda,\mathcal{A},\mathcal{H},\mu)} = \int [f, g]_{\mathcal{H}}\, \mathrm{d}\mu$.

The two following special instances of the previous example will be of particular interest.

(5) Let $(\Lambda, \mathcal{A})$ be a measurable space and μ a finite non-negative measure on $(\Lambda, \mathcal{A})$. Then $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ is a normal $\mathcal{L}_b(\mathcal{G}_0)$-Hilbert module for the module action $(\mathrm{P} \bullet \Phi)(\cdot) = \mathrm{P}\Phi(\cdot)$ and gramian $[\Phi, \Psi]_{L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)} = \int \Phi\Psi^H\, \mathrm{d}\mu$ for $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0)$. Note that in this case

$$\langle \Phi, \Psi\rangle_{L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)} = \mathrm{Tr}[\Phi, \Psi]_{L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)} = \int \mathrm{Tr}\left(\Phi\Psi^H\right) \mathrm{d}\mu ,$$

and the associated squared norm is

$$\|\Phi\|^2_{L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)} = \int \mathrm{Tr}\left(\Phi\Phi^H\right) \mathrm{d}\mu = \int \left\|\Phi\Phi^H\right\|_1 \mathrm{d}\mu = \int \|\Phi\|_2^2\, \mathrm{d}\mu .$$

(6) Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Then the space* $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ *of centered variables in* $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ *is a normal Hilbert* $\mathcal{L}_b(\mathcal{H}_0)$*-module for the module action* $\mathrm{P} \bullet Y = \mathrm{P}Y$ *and gramian* $[Y, Z]_{\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})} = \mathrm{Cov}\,(Y, Z) = \mathbb{E}\,[Y \otimes Z]$ *for* $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$*, and* $Y, Z \in \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$*.*

In the univariate case, the measure $\hat{X}$ obtained by Theorem 1.2 is valued in the space of centered $L^2(\Omega, \mathcal{F}, \mathbb{P})$ variables and is orthogonally scattered. In the Hilbert valued case, it is valued in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, which is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module. In the following section, we thus extend c.a.o.s. measures to replace the standard orthogonality property by the more general gramian-orthogonality property.

2.4 Countably additive gramian orthogonally scattered measures

This section aims at presenting the generalization of c.a.o.s. measures to normal Hilbert modules. Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and $(\Lambda, \mathcal{A})$ a measurable space. Let ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. A c.a.g.o.s. measure W on $(\Lambda, \mathcal{A}, \mathcal{H})$ with intensity operator measure ν is a mapping $W : \mathcal{A} \to \mathcal{H}$ such that, for all $A, B \in \mathcal{A}$, $[W(A), W(B)]_{\mathcal{H}} = \nu(A \cap B)$. In fact, the intensity operator measure ν can be deduced from W as in the following definition.

Definition 2.8 (c.a.g.o.s. measure)**.** *Let* $\mathcal{H}_0$ *be a separable Hilbert space,* $\mathcal{H}$ *a normal Hilbert* $\mathcal{L}_b(\mathcal{H}_0)$*-module and* $(\Lambda, \mathcal{A})$ *a measurable space. We say that* $W : \mathcal{A} \to \mathcal{H}$ *is a* countably additive gramian-orthogonally scattered *(c.a.g.o.s.) measure on* $(\Lambda, \mathcal{A}, \mathcal{H})$ *if it is a* $\mathcal{H}$*-valued measure on* $(\Lambda, \mathcal{A})$ *such that for all* $A, B \in \mathcal{A}$*,*

$$A \cap B = \emptyset \Rightarrow [W(A), W(B)]_{\mathcal{H}} = 0 \; .$$

In this case, the mapping

$$\nu_W : A \mapsto [W(A), W(A)]_{\mathcal{H}}$$

is a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ *called the* intensity operator measure *of* W *and we have that, for all* $A, B \in \mathcal{A}$*,*

$$\nu_W(A \cap B) = [W(A), W(B)]_{\mathcal{H}} \; .$$

A special case that will be of interest for us is the following.

Definition 2.9 (Random c.a.g.o.s. measure)**.** *Let* $\mathcal{H}_0$ *be a separable Hilbert space,* $(\Lambda, \mathcal{A})$ *be a measurable space, and* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A c.a.g.o.s. measure on the space* $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ *is called an* $\mathcal{H}_0$-valued random *c.a.g.o.s. measure defined on* $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$*.*

It is straightforward to see that a c.a.g.o.s. measure W is a c.a.o.s. measure with intensity measure $\|\nu_W\|_1$ which, in particular implies that, when Λ is a locally-compact topological space, W is regular if and only if $\|\nu_W\|_1$ is regular. By the known integration theory for c.a.o.s. measures (see Appendix B.2), it is possible to integrate scalar-valued functions of $L^2(\Lambda, \mathcal{A}, \|\nu_W\|_1)$ with respect to W, simply by an isometry extension of the mapping $\mathbb{1}_A \mapsto W(A)$, $A \in \mathcal{A}$. In Section 2.8, we will generalize this approach to define the integral of an operator-valued function with respect to a random c.a.g.o.s. measure W, this time by a gramian isometry extension of the mapping $\mathbb{1}_A\mathrm{P} \mapsto \mathrm{P}\,W(A)$, $A \in \mathcal{A}$, $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$. Before that, we build the normal Hilbert module generated by $\mathbb{1}_A\mathrm{P}$, $A \in \mathcal{A}$, $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$. This turns out to be me more involved in the infinite dimensional case as it requires two steps. First, in Section 2.5, we will define the space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ of bounded-operator-valued functions which are *square-integrable* with respect to ν. Then, because this space is not complete, except in very special cases investigated in Appendix B.3, we explain in Section 2.6 how to complete this space by adding function with values that are possibly non-bounded operators. Having this complete space at hand allows us to introduce interesting gramian-isometries on such spaces, namely, in Section 2.7 by right-composition by an operator-valued function, in Section 2.8 by integration with respect to a random c.a.g.o.s. measure and we conclude in Section 2.9 by investigating how to define the filtering of a random c.a.g.o.s. measure.

2.5 The space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda, \mathcal{A})$ be a measurable space and ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. The goal of this section is to introduce the space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ of bounded-operator-valued functions which are *square-integrable* with respect to ν. This space includes the space $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$, and the inclusion is, in general, strict. More precisely, let $\mathrm{P}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, then it is easy to check that $A \mapsto \mathrm{P}\nu(A)\mathrm{Q}^{\mathsf{H}}$ defines a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure. By linearity, such a definition can be extended to the case where Φ, Ψ are simple functions from Λ to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and it is then natural to want to provide a meaning to an integral of the type $\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}}$ where $\Phi, \Psi \in \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ or, more generally, in $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. Since ν has a density with respect to any measure μ dominating $\|\nu\|_1$, the construction of such integrals is very similar to the work done in [62] but is more general as discussed in Section 6.3. We follow [33, 45] in this approach, which is a natural extension to the finite dimensional case investigated in [54].

Definition 2.10. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with density $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Let $\Phi, \Psi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, then the pair (Φ, Ψ) is said to be ν-integrable if $\Phi f \Psi^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$ and in this case we define*

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int \Phi f \Psi^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{G}_0) \; . \tag{2.7}$$

If (Φ, Φ) is ν-integrable we say that Φ is square ν-integrable and we denote by $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ the space of functions in $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ which are square ν-integrable.

To check that Φ is square ν-integrable, we can replace $\|\nu\|_1$ by an arbitrary dominating measure μ (often taken as Lebesgue's measure, as in [62]), as stated in the following result.

Proposition 2.7. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let μ be a finite non-negative measure on $(\Lambda, \mathcal{A})$ which dominates $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Let $\Phi, \Psi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. Then (Φ, Ψ) is ν-integrable if and only if $\Phi g \Psi^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \mu)$, and, in this case, we have*

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int \Phi g \Psi^{\mathsf{H}} \, \mathrm{d}\mu \; . \tag{2.8}$$

Moreover, we have

$$\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) \Leftrightarrow \Phi g^{1/2} \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu) \; , \tag{2.9}$$

and, if $\Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then (Φ, Ψ) is ν-integrable and

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \, \mathrm{d}\mu. \tag{2.10}$$

The equivalence in (2.9) says that $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is the preimage of $\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ by the mapping

$$\begin{array}{lcl} \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)) & \to & \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)) \\ \Phi & \mapsto & \Phi g^{1/2} \end{array}$$

and (2.10) can be rewritten as

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \left[\Phi g^{1/2}, \Psi g^{1/2}\right]_{\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)}$$

where $[\cdot, \cdot]_{\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)}$ is the pseudo-gramian (in the sense that is satisfies all the conditions of Definition 2.4 except (ii)) defined on $\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$. This pseudo-gramian becomes a gramian on $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ which we recall is obtained by quotienting $\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ with the μ-a.e. equality and this new space is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module, see Example 2.1(5). This naturally leads to the following proposition.

Proposition 2.8. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is a $\mathcal{L}_b(\mathcal{G}_0)$-module with module action*

$$\mathrm{P} \bullet \Phi : \lambda \mapsto \mathrm{P}\Phi(\lambda), \quad \mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0), \Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$$

and the relation

$$[\Phi, \Psi]_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} \quad \Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) \tag{2.11}$$

is a pseudo-gramian on $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and a gramian on the quotient space

$$\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) := \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) \Big/ \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\} .$$

Moreover $\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu), [\cdot, \cdot]_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)}\right)$ is a normal pre-Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module and, for any finite non-negative measure μ dominating $\|\nu\|_1$ with density $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$,

$$\left\{\Phi \,:\, \Phi g^{1/2} = 0 \quad \mu\text{-a.e.}\right\} = \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\} , \tag{2.12}$$

and the mapping $\Phi \mapsto \Phi g^{1/2}$ is a gramian-isometry from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ to $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$.

2.6 Completion of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$

In the multivariate case (*i.e.* when $\mathcal{H}_0$ and $\mathcal{G}_0$ have finite dimensions) the completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is proven in [54]. However completeness is not guaranteed in the infinite dimensional case, see [45], where the authors refer to [41] for a counter-example. In Appendix B.3, we pursue this line of thoughts by providing a necessary and sufficient condition for the completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ in the general case. Since the integral of operator-valued functions with respect to a c.a.g.o.s. measure is expected to be a gramian-unitary operator, it must be defined on a complete space. A first option is then to complete the space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ by taking the equivalence classes of Cauchy sequences such that two such sequences (U_n) and (V_n) are in the same class if $\lim(U_n - V_n) = 0$. However, the completed space is very abstract and hard to describe in an intuitive way. More concretely the uncompleteness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ comes from the fact that we restrict ourselves to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions. A more concrete complete extension of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$, as noticed in [33, Section 3.4] and [45], simply consists in extending this space to include well chosen $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$-valued functions. We now follow their path.

Definition 2.11. *Let $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces, ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Let $\Phi, \Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, then the pair (Φ, Ψ) is said to be ν-integrable if the three following assertions hold.*

(i) $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Phi)$ *and* $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Psi)$ $\|\nu\|_1$*-a.e.*

(ii) $\Phi f^{1/2}$ *and* $\Psi f^{1/2}$ *are* $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$*-valued.*

(iii) $(\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$.

In this is the case, we define for all $A \in \mathcal{A}$,

$$\int_A \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int_A (\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{G}_0) . \tag{2.13}$$

If (Φ, Φ) is ν-integrable, then Φ is said to be square ν-integrable and we denote by $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ the space of functions in $\mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ which are square ν-integrable.

Note that, if Φ and Ψ are $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued, we can write $(\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} = \Phi f \Psi^{\mathsf{H}}$ because the adjoint of Ψ exists. In the general case the latter exists only when $\mathcal{D}(\Psi)$ is dense in $\mathcal{H}_0$. The left hand side term of (2.13) should therefore be taken only as a shorthand notation for the right hand side term which makes sense because of (ii). As previously, we can show that $\|\nu\|_1$ can be replaced by any finite non-negative measure μ dominating $\|\nu\|_1$ and the following characterization holds.

Proposition 2.9. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let μ be a finite non-negative measure dominating $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Let $\Phi, \Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, then (Φ, Ψ) is ν-integrable if and only if it satisfies*

(i') $\mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Phi)$ *and* $\mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Psi)$ *μ-a.e.*

(ii') $\Phi g^{1/2}$ *and* $\Psi g^{1/2}$ *are* $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$*-valued.*

(iii') $(\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \mu)$.

In this case we have for all $A \in \mathcal{A}$,

$$\int_A \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int_A (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \, \mathrm{d}\mu \,. \tag{2.14}$$

Moreover, we have

$$\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \Leftrightarrow \begin{cases} \mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Phi) \ \mu\text{-a.e.} \\ \Phi g^{1/2} \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu) \end{cases} \tag{2.15}$$

and, if $\Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then (Φ, Ψ) is ν-integrable and

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \, \mathrm{d}\mu = \left[\Phi g^{1/2}, \Psi g^{1/2}\right]_{\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} . \tag{2.16}$$

Similarly as before, we get the following (more general) result.

Theorem 2.10. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is a $\mathcal{L}_b(\mathcal{G}_0)$-module with module action*

$$\mathrm{P} \bullet \Phi : \lambda \mapsto \mathrm{P}\Phi(\lambda), \quad \mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0), \Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$$

and the relation

$$[\Phi, \Psi]_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)} := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} \quad \Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \,, \tag{2.17}$$

is a pseudo-gramian on $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and a gramian on the quotient space

$$\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) := \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \Big/ \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\} .$$

Moreover, $\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu), [\cdot, \cdot]_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)}\right)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module and, for any finite non-negative measure μ dominating $\|\nu\|_1$ with density $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then

$$\left\{\Phi \,:\, \Phi g^{1/2} = 0 \quad \mu\text{-a.e.}\right\} = \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\} \,, \tag{2.18}$$

and the mapping $\Phi \mapsto \Phi g^{1/2}$ is a gramian unitary operator from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ to $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$.

Given a trace-class measure ν on $(\Lambda, \mathcal{A})$, we now have three different spaces of *square integrable operator-valued* functions: the usual $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ space with $\|\nu\|_1$-integrated squared operator norm, the space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ of Definition 2.10 and the space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ of Definition 2.11. It is easily seen that they verify the inclusions

$$L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) \subset \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \,, \tag{2.19}$$

where the second inclusion is an isometric embedding and the first one a continuous embedding. More precisely, if $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then

$$\|\Phi\|_{L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)} \geq \|\Phi\|_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} = \|\Phi\|_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)} \,, \tag{2.20}$$

with the convention that $\|\Phi\|_{L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)} = \infty$ if $\Phi \notin L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$.

We conclude this section by the following theorem stating that $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ can be recovered by density of well chosen subspaces.

Theorem 2.11. *Let $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Then the space $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. In particular, this implies the two following assertions.*

(i) The space $\mathrm{Span}\left(\mathbb{1}_A \mathrm{P} \,:\, A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)\right)$ of simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.

(ii) In the case where $\Lambda = \hat{\mathbb{G}}$ and $\mathcal{A} = \mathcal{B}(\hat{\mathbb{G}})$ with $\mathbb{G}$ being an l.c.a. group, we get that the space $\mathrm{Span}\left(\mathrm{e}_t \mathrm{P} \,:\, t \in \mathbb{G}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)\right)$ is dense in $\mathsf{L}^2(\mathbb{G}, \mathcal{B}(\mathbb{G}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ where $\mathrm{e}_t : \chi \mapsto \chi(t)$.

2.7 Pointwise composition of operator valued functions

Let $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ be separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. For $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{O}(\mathcal{G}_0, \mathcal{I}_0)$ valued functions Φ and Ψ, we denote by $\Psi\Phi$ the pointwise composition, that is $\Psi\Phi : \lambda \mapsto \Psi(\lambda) \circ \Phi(\lambda)$. Of course we need the image of $\Phi(\lambda)$ to be included in the domain of $\Psi(\lambda)$ for this pointwise composition to be well defined. To check whether $\Psi\Phi$ is square integrable with respect to ν, we can equivalently check that Ψ is square integrable with respect to the trace class p.o.v.m. $\Phi\nu\Phi^{\mathsf{H}}$ defined by

$$\Phi\nu\Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu \Phi^{\mathsf{H}} ,$$

which is a well defined trace-class p.o.v.m. whenever $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. The space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ is then characterized by the following theorem.

Theorem 2.12. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $\Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{I}_0)$. Then*

$$\Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) \Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu) . \tag{2.21}$$

Moreover, the following assertions hold.

(a) For all $\Psi, \Theta \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$,

$$(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}} = \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}} .$$

(b) The mapping $\Psi \mapsto \Psi\Phi$ is a well defined gramian-isometry from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ to $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)$.

(c) Suppose moreover that Φ is injective $\|\nu\|_1$-a.e., then we have that

$$\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}}) ,$$

where we define $\Phi^{-1}(\lambda) := \left(\Phi(\lambda)_{|\mathcal{D}(\Phi(\lambda)) \to \mathrm{Im}(\Phi(\lambda))}\right)^{-1}$ with domain $\mathrm{Im}(\Phi(\lambda))$ for all $\lambda \in \{\Phi \text{ is injective}\}$ and $\Phi^{-1}(\lambda) = 0$ otherwise.

2.8 Integration with respect to a random c.a.g.o.s. measure

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda, \mathcal{A})$ be a measurable space, and let ν be a trace-class p.o.v.m. defined on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. We denote

$$\widehat{\mathcal{H}}^{\nu, \mathcal{G}_0} := \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) , \tag{2.22}$$

which is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module by Theorem 2.10. Given an $\mathcal{H}_0$-valued random c.a.g.o.s. measure W, we further denote

$$\mathcal{H}^{W, \mathcal{G}_0} := \overline{\mathrm{Span}}^{\mathcal{G}} \left(\mathrm{P}W(A) \,:\, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0),\, A \in \mathcal{A}\right) , \tag{2.23}$$

which is a submodule of $\mathcal{G} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$.

We can now define the integral of an $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$-valued function with respect to a random c.a.g.o.s. measure W with intensity operator measure ν_W through a gramian isometry defined on $\widehat{\mathcal{H}}^{\nu, \mathcal{G}_0}$.

Theorem 2.13. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let $\widehat{\mathcal{H}}^{\nu_W, \mathcal{G}_0}$ and $\mathcal{H}^{W, \mathcal{G}_0}$ be defined as in (2.22) and (2.23). Then there exists a unique gramian isometry*

$$I_W^{\mathcal{G}_0} : \widehat{\mathcal{H}}^{\nu_W, \mathcal{G}_0} \to \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$$

such that, for all $A \in \mathcal{A}$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$,

$$I_W^{\mathcal{G}_0}(\mathbb{1}_A \mathrm{P}) = \mathrm{P} W(A) \quad \mathbb{P}\text{-a.s.}$$

Moreover, $\widehat{\mathcal{H}}^{\nu_W, \mathcal{G}_0}$ and $\mathcal{H}^{W, \mathcal{G}_0}$ are gramian-isometrically isomorphic.

Taking $\mathcal{G}_0 = \mathbb{C}$ in the previous result, the Hilbert modules $\widehat{\mathcal{H}}^{\nu_W, \mathbb{C}}$ and $\mathcal{H}^{W, \mathbb{C}}$ reduce to Hilbert spaces and their gramians to scalar products as in Example 2.1(1), so that the gramian isomorphism between them also reduces to a standard Hilbert isomorphism.

We can now define the integral of an operator valued function with respect to W.

Definition 2.12 (Integral with respect to a random c.a.g.o.s. measure)**.** *Under the assumptions of Theorem 2.13, we use an integral sign to denote $I_W^{\mathcal{G}_0}(\Phi)$ for $\Phi \in \widehat{\mathcal{H}}^{\nu_W, \mathcal{G}_0}$. Namely, we write*

$$\int \Phi \, \mathrm{d}W = \int \Phi(\lambda) \, W(\mathrm{d}\lambda) := I_W^{\mathcal{G}_0}(\Phi) \, . \tag{2.24}$$

The following results shows that we can restrict ourselves to $\mathcal{G}_0 = \mathbb{C}$ to characterize the c.a.g.o.s. W through the isometry $I_W^{\mathbb{C}}$, in which case, as previously mentioned, the gramian isometry is a standard isometry between two Hilbert spaces.

Theorem 2.14. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $\mathcal{H}_0$ be a separable Hilbert space and ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Define $\widehat{\mathcal{H}}^{\nu, \mathbb{C}}$ as in (2.23). Then for any isometry $w : \widehat{\mathcal{H}}^{\nu, \mathbb{C}} \to \mathcal{M}(\Omega, \mathcal{F}, \mathbb{P})$, there exists a unique $\mathcal{H}_0$-valued random c.a.g.o.s. measure W on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν such that, for all $\varphi \in \widehat{\mathcal{H}}^{\nu, \mathbb{C}}$,*

$$w(\varphi) = \int \varphi \, \mathrm{d}W \quad \mathbb{P}\text{-a.s.} \tag{2.25}$$

2.9 Filtering random c.a.g.o.s. measures

If $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ as in Definition 2.11, then it is immediate to check that for any $A \in \mathcal{A}$, $\mathbb{1}_A \Phi : \lambda \mapsto \mathbb{1}_A(\lambda)\Phi(\lambda)$ also belongs to $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Then, applying the definition of $\Phi \nu \Phi^{\mathsf{H}}$ in Section 2.7 and the integral of Definition 2.12, we get the following result.

Corollary 2.15. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Then the mapping*

$$V : A \mapsto \int_A \Phi \, \mathrm{d}W = I_W^{\mathcal{G}_0}(\mathbb{1}_A \Phi) \tag{2.26}$$

is a $\mathcal{G}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure $\Phi \nu_W \Phi^{\mathsf{H}}$.

We can interpret (2.26) as saying that V admits the density Φ with respect to W, and write $\mathrm{d}V = \Phi \mathrm{d}W$ (or, equivalently, $V(\mathrm{d}\lambda) = \Phi(\lambda) W(\mathrm{d}\lambda)$). In the following we will rather use a signal processing interpretation where Λ is seen as a set of frequencies and Φ is seen as a transfer operator function acting on the (random) input frequency distribution W.

Definition 2.13 (Filter $\hat{F}_\Phi(W)$ acting on a random c.a.g.o.s. measure in $\hat{\mathcal{S}}_\Phi$)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. For a given transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, we denote by $\hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ the set of $\mathcal{H}_0$-valued random c.a.g.o.s. measures on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ whose intensity operator measures ν_W satisfy $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Then, for any $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$, we say that the random $\mathcal{G}_0$-valued c.a.g.o.s. measure V defined by (2.26) is the output of the filter with transfer operator function Φ applied to the input c.a.g.o.s. measure W, and we denote $V = \hat{F}_\Phi(W)$.*

The following corollary is obtained from Theorem 2.12 and allows us to deal with the composition and inversion of filters on random c.a.g.o.s. measures.

Corollary 2.16 (Composition and inversion of filters on random c.a.g.o.s. measures)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces, and $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. Let $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Then three following assertions hold.*

(i) For any separable Hilbert space $\mathcal{I}_0$, we have

$$I^{\mathcal{I}_0}_{\hat{F}_\Phi(W)}\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu_W\Phi^{\mathsf{H}})\right) \underset{\sim}{\subsetneq} I^{\mathcal{I}_0}_W\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu_W)\right) . \tag{2.27}$$

(ii) For any separable Hilbert space $\mathcal{I}_0$ and all $\Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{I}_0)$, we have $W \in \hat{\mathcal{S}}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$, and in this case, we have

$$\hat{F}_\Psi \circ \hat{F}_\Phi(W) = \hat{F}_{\Psi\Phi}(W). \tag{2.28}$$

(iii) Suppose that Φ is injective $\|\nu_W\|_1$-a.e. Then $W = F_{\Phi^{-1}} \circ F_\Phi(W)$, where Φ^{-1} is defined as in Assertion (c) of Theorem 2.12. Moreover, Assertion (i) above holds with $\underset{\sim}{\subsetneq}$ replaced by $\cong$ in Equation (2.27).

Note that, using the notation in (2.23), the two spaces on the left-hand and right-hand sides of (2.27) can be compactly written as $\mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0}$ and $\mathcal{H}^{W, \mathcal{I}_0}$, respectively.

3 Application to Hilbert valued weakly-stationary processes

Now, we have all the tools to derive spectral representations of Hilbert valued weakly-stationary processes. We follow [33, Section 4.2] and then derive linear filtering of such processes based on the spectral representation thereby constructed.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ a separable Hilbert space and $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Let $X = (X_t)_{t \in \mathbb{G}} \in \mathcal{H}^{\mathbb{G}}$ be a centered, weakly-stationary, $\mathcal{H}_0$-valued process indexed by an l.c.a. group $(\mathbb{G}, +)$. By analogy to the univariate case, and taking into account the module structure of $\mathcal{H}$, let us define the *modular time domain* of X as the submodule of $\mathcal{H}$ generated by the X_t's, that is

$$\mathcal{H}^X := \overline{\mathrm{Span}}^{\mathcal{H}}\left(\mathrm{P}X_t \;:\; \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0),\, t \in \mathbb{G}\right) .$$

For all $h \in \mathbb{G}$, define (using Proposition 2.5) the shift operator of lag h as the unique gramian-unitary operator $U^X_h : \mathcal{H}^X \to \mathcal{H}^X$ which maps X_t to X_{t+h} for all $t \in \mathbb{G}$. As in the univariate case (see Lemma 1.1), weak stationarity is characterized by the representation properties of U^X seen in Definition 2.7.

Lemma 3.1. *Let $X := (X_t)_{t \in \mathbb{G}}$ be a centered, L^2, $\mathcal{H}_0$-valued process. Then X is weakly stationary if and only if U^X is a c.g.u.r. of $\mathbb{G}$ on $\mathcal{H}^X$.*

In particular (see also Remark 1.4), continuity of U^X is equivalent to weak-continuity of Γ_X from Definition 1.3 (see [33, Proposition 2.5.2]). The following theorem (see [33, Theorem 4.2.2, Theorem 4.2.4]) gives **R1**, **R2**.

Theorem 3.2. *Let $\mathcal{H}_0$ be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbb{G}, +)$ be an l.c.a. group. Let $X := (X_t)_{t \in \mathbb{G}}$ be a centered, L^2, $\mathcal{H}_0$-valued process. Then X is weakly stationary if and only if there exists a regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \Omega, \mathcal{F}, \mathbb{P})$ such that*

$$X_t = \int \mathrm{e}_t \, \mathrm{d}\hat{X} = \int \chi(t) \, \hat{X}(\mathrm{d}\chi) \quad \text{for all} \quad t \in \mathbb{G} . \tag{3.1}$$

In this case, $\hat{X}$ is uniquely determined by (3.1) *and is called the* spectral representation *of X. The intensity operator measure ν_X of $\hat{X}$ is called the* spectral operator measure *of X. It is a regular trace-class p.o.v.m. on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{H}_0)$ and is the unique regular p.o.v.m. satisfying*

$$\Gamma_X(h) = \int \mathrm{e}_h \, \mathrm{d}\nu_X = \int \chi(h) \, \nu_X(\mathrm{d}\chi) \quad \text{for all} \quad h \in \mathbb{G} . \tag{3.2}$$

Note that, like for Bochner's theorem in the univariate case, Relation (3.2) can be obtained without using stochastic processes and this result can also be used to derive spectral analysis for weakly-stationary stochastic processes. This will be discussed in Section 6.

The second part of Theorem 3.2 directly provides an answer to the comments made in Remark 1.3 about **R1**. We now answer the comments made in the same remark about **R2**, that is, using Theorem 3.2 with the definitions and results of Section 2.6 and Section 2.8, we can exhibit the isomorphism that relates the *time domain* with the *spectral domain*. For sake of generality, we first extend the definition of the time domain to the case where the $\mathcal{H}_0$-valued process X is used to generate variables valued in a possibly different Hilbert space $\mathcal{G}_0$.

(i) Define the $\mathcal{G}_0$-valued *time domain* of X by

$$\mathcal{H}^{X,\mathcal{G}_0} := \overline{\mathrm{Span}}^{\mathcal{M}(\Omega,\mathcal{F},\mathcal{G}_0,\mathbb{P})}\left(\mathrm{P}X_t \ : \ \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\, t \in \mathbb{G}\right) ,$$

which is a submodule of $\mathcal{M}(\Omega,\mathcal{F},\mathcal{G}_0,\mathbb{P})$.

(ii) Define the $\mathcal{G}_0$-valued *spectral domain* of X by

$$\widehat{\mathcal{H}}^{X,\mathcal{G}_0} := \mathsf{L}^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_X) ,$$

which is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module by Theorem 2.10.

Then, for any separable Hilbert space $\mathcal{G}_0$, the mapping $\Phi \mapsto \int \Phi \mathrm{d}\hat{X}$ is a gramian unitary operator from $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ to $\mathcal{H}^{X,\mathcal{G}_0}$, and thus it makes the $\mathcal{G}_0$-valued *time domain* $\mathcal{H}^{X,\mathcal{G}_0}$ and the $\mathcal{G}_0$-valued *spectral domain* $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ *gramian-isometrically isomorphic*, as announced in Remark 1.3.

Remark 3.1. *We introduced the notation $\mathcal{H}^{X,\mathcal{G}_0}$ and $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ above to stress that these spaces are derived from a given centered weakly stationary process X. Of course they are related to the notation $\widehat{\mathcal{H}}^{\nu,\mathcal{G}_0}$ and $\mathcal{H}^{W,\mathcal{G}_0}$ introduced in (2.22) and (2.23), respectively for a p.o.v.m. ν and a c.a.g.o.s. W. Namely, we have $\mathcal{H}^{X,\mathcal{G}_0} = \mathcal{H}^{\hat{X},\mathcal{G}_0}$ and $\widehat{\mathcal{H}}^{X,\mathcal{G}_0} = \widehat{\mathcal{H}}^{\nu_X,\mathcal{G}_0}$.*

We now answer Remark 1.3 about **R3**, and in particular the two points a) and b) raised in this remark. As explained in Remark 1.3, the transfer operator function of a linear shift-invariant filter with input given by an $\mathcal{H}_0$-valued weakly stationary time series $(X_t)_{t\in\mathbb{G}}$ and a well defined $\mathcal{G}_0$-valued output $(Y_t)_{t\in\mathbb{G}}$ corresponds to the spectral representation $\Phi \in \widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ of $Y_0 \in \mathcal{H}^{X,\mathcal{G}_0}$. In other words, the filter with transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\mathcal{H}_0,\mathcal{G}_0\right)$ applies to the $\mathcal{H}_0$-valued weakly stationary time series $X = (X_t)_{t\in\mathbb{G}}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ if and only if

$$\hat{X} \in \hat{\mathcal{S}}_\Phi(\Omega,\mathcal{F},\mathbb{P}) \quad \text{or, equivalently,} \quad \Phi \in \widehat{\mathcal{H}}^{X,\mathcal{G}_0} , \tag{3.3}$$

where $\hat{\mathcal{S}}_\Phi(\Omega,\mathcal{F},\mathbb{P})$ is as in Definition 2.13. Then the output $Y = (Y_t)_{t\in\mathbb{G}}$ is defined by its spectral random c.a.g.o.s.measure $\hat{Y} = \hat{F}_\Phi(\hat{X})$. For convenience we write, in the time domain,

$$X \in \mathcal{S}_\Phi(\Omega,\mathcal{F},\mathbb{P}) \quad \text{and} \quad Y = F_\Phi(X) , \tag{3.4}$$

for the assertions $\hat{X} \in \hat{\mathcal{S}}_\Phi(\Omega,\mathcal{F},\mathbb{P})$ and $\hat{Y} = \hat{F}_\Phi(\hat{X})$. Condition (3.3) answers Question a) of Remark 1.3 about **R3**. Then we can answer Question b), by simply transposing Corollary 2.16 in the time domain as follows.

Proposition 3.3 (Composition and inversion of filters on weakly stationary time series)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and pick a transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\mathcal{H}_0,\mathcal{G}_0\right)$. Let X be a centered weakly stationary $\mathcal{H}_0$-valued process defined on $(\Omega,\mathcal{F},\mathbb{P})$ with spectral operator measure ν_X. Suppose that $X \in \mathcal{S}_\Phi(\Omega,\mathcal{F},\mathbb{P})$ and set $Y = F_\Phi(X)$, as defined in (3.4). Then the three following assertions hold.*

(i) For any separable Hilbert space $\mathcal{I}_0$, we have $\mathcal{H}^{Y,\mathcal{I}_0} \precsim \mathcal{H}^{X,\mathcal{I}_0}$.

(ii) For any separable Hilbert space $\mathcal{I}_0$ and all $\Psi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),\mathcal{G}_0,\mathcal{I}_0\right)$, we have $X \in \mathcal{S}_{\Psi\Phi}(\Omega,\mathcal{F},\mathbb{P})$ if and only if $Y = F_\Phi(X) \in \mathcal{S}_\Psi(\Omega,\mathcal{F},\mathbb{P})$, and in this case, we have

$$F_\Psi \circ F_\Phi(X) = F_{\Psi\Phi}(X). \tag{3.5}$$

(iii) Suppose that Φ is injective $\|\nu_X\|_1$-a.e. Then $X = F_{\Phi^{-1}} \circ F_\Phi(X)$, where Φ^{-1} is defined as in Assertion (c) of Theorem 2.12. Moreover, Assertion (i) above holds with $\precsim$ replaced by $\cong$.

4 Statistical inference

In the following we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a functional time series $(X_t)_{t\in\mathbb{Z}}$ valued in a separable real Hilbert space $\mathcal{H}_0$. Statistical results about time series rely on dependence assumptions between the observations $(X_t)_{t\in\mathbb{Z}}$. Intuitively, we assume that the observations are dependent but not too much. For example, given $m \in \mathbb{N}^*$, we say that $(X_t)_{t\in\mathbb{Z}}$ is m-dependent if for all $t \in \mathbb{Z}$ the σ-fields $\mathcal{F}_{t+m}^+$ and $\mathcal{F}_t^-$ are independent, where for all $t \in \mathbb{Z}$,

$$\mathcal{F}_t^- := \sigma(X_s, s \le t) \quad \text{and} \quad \mathcal{F}_t^+ := \sigma(X_s, s \ge t)\ .$$

This notion is however still too restricitve for time series analysis and several assumptions can be made to relax m-dependence. We present the three most important : mixing conditions, assumptions based on shifted causal representations and assumptions based on cumulant conditions. We mainly focus on the two last because of their use in recent works on spectral analysis for functional time series (*e.g.* [28, 39, 29, 37, 15, 62]).

4.1 The main dependence assumptions

4.1.1 General mixing conditions

Mixing conditions are based on measuring the dependence between disjoint groups of observations taken from the time series $(X_t)_{t\in\mathbb{Z}}$. There exists a multitude of coefficients to measure such a dependence and one of the most popular is the α-mixing coefficient. A strictly stationary time series $(X_t)_{t\in\mathbb{Z}}$ is said to be *α-mixing* (or *strongly mixing*) if

$$\alpha(n) := \sup\left\{|P(A\cap B) - P(A)P(B)|\ :\ A \in \mathcal{F}_0^-, B \in \mathcal{F}_n^+\right\} \xrightarrow[n\to+\infty]{} 0\ .$$

This definition does not depend on the space where X_t is valued and can therefore be used for functional time series. Mixing conditions have had an important role on the proof of limit theorems such as the Central Limit Theorem or convergence of partial sums in the Skohorod space are proven for dependent univariate processes (see *e.g.* [20, 21, 7]) but also on spectral estimation [55].

4.1.2 Using shifted causal representations (SCRs)

We say that a time series $(X_t)_{t\in\mathbb{Z}}$ admits a shifted causal representation (SCR) with respect to the i.i.d. sequence $(\epsilon_t)_{t\in\mathbb{Z}}$ if there exists a measurable function g such that for all $t \in \mathbb{Z}$,

$$X_t = g(\epsilon_t, \epsilon_{t-1}, \cdots)\ .$$

This representation generalizes the standard linear model to non-linear dependence on the sequence $(\epsilon_t)_{t\in\mathbb{Z}}$. Note that, since $(\epsilon_t)_{t\in\mathbb{Z}}$ is i.i.d, $(X_t)_{t\in\mathbb{Z}}$ is strictly stationnary. One can then control the dependence structure of $(X_t)_{t\in\mathbb{Z}}$ by making assumption on its distance to a process obtained by replacing one or more of the ϵ_t's by independent copies. Following this idea, several asymptotic results were derived for univariate time series (see *e.g.* [59, 43]). For functional time series the main assumption found in the literature based on SCRs, is L^p-m-approximability as introduced in [28].

Definition 4.1. *Let $p, m \in \mathbb{N}^*$, and consider a sequence $(X_t)_{t\in\mathbb{Z}} \subset L^p(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ admitting the SCR $X_t = g(\epsilon_t, \epsilon_{t-1}, \cdots)$ with respect to the i.i.d. sequence $(\epsilon_t)_{t\in\mathbb{Z}}$. It is said to be L^p-m-approximable if, for all independent copy $(\tilde{\epsilon}_t)_{t\in\mathbb{Z}}$ of $(\epsilon_t)_{t\in\mathbb{Z}}$, we have*

$$\sum_{m=1}^{+\infty} \left\| X_0 - \tilde{X}_0^{(m)} \right\|_{L^p(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})} < +\infty \tag{4.1}$$

where $\tilde{X}_t^{(m)} = g(\epsilon_t, \epsilon_{t-1}, \cdots, \epsilon_{t-m+1}, \tilde{\epsilon}_{t-m}, \tilde{\epsilon}_{t-m-1}, \cdots)\ .$

As explained in [28], condition (4.1), implies that $(X_t)_{t\in\mathbb{Z}}$ can be approximated in L^p-norm by an m-dependent time series. Note that if $q \ge p$, then L^q-m-approximability implies L^p-m-approximability.

Recently, the following assumption has also been proposed in [15] for some $p \in \mathbb{N}^*$,

Assumption 4.1 (p). *Let $(X_t)_{t\in\mathbb{N}}$ admit the SCR $X_t = g(\epsilon_t, \epsilon_{t-1}, \cdots)$, and suppose that for all $t \in \mathbb{Z}$, $X_t \in L^p(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, and for all independent copy $(\tilde{\epsilon}_t)_{t\in\mathbb{Z}}$ of $(\epsilon_t)_{t\in\mathbb{Z}}$, we have*

$$\sum_{t=0}^{+\infty} \| X_t - \mathbb{E}\left[X_t | \mathcal{G}_t \right] \|_{L^p(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})} < +\infty \tag{4.2}$$

where $\mathcal{G}_t = \sigma(\epsilon_t, \cdots, \epsilon_1, \tilde{\epsilon}_0, \epsilon_{-1}, \cdots)$.

4.1.3 Using cumulants

Let $k \in \mathbb{N}^*$, denote by $\mathcal{V}(k)$ the set of all partitions of $\{1, \cdots, k\}$ and let $Y_1, \cdots, Y_k \in L^k(\Omega, \mathcal{F}, \mathbb{P})$. Then the *cumulant* of $(Y_1, \cdots, Y_k)$ is defined by

$$\mathrm{cum}(Y_1, \cdots, Y_k) = \sum_{\nu=(\nu_1,\cdots,\nu_p)\in\mathcal{V}(k)} (-1)^{p-1}(p-1)! \prod_{\ell=1}^{p} \mathbb{E}\left[\prod_{j\in\nu_\ell} Y_j \right] .$$

The cumulant can be seen as a generalization of the covariance to orders larger than 2, in fact we see that, when $k = 2$ and $Y_1, Y_2 \in \mathbb{R}$, then $\mathrm{cum}(Y_1, Y_2) = \mathrm{Cov}\,(Y_1, Y_2)$.

In order to generalize this notion for functional time series, we cannot use a similar, direct expression, since we cannot take the product of $\mathcal{H}_0$-valued variables. However, using the notion of tensor product of Hilbert spaces one can define the cumulant of k functional random variables similarly to the covariance operator. In short, [32, Theorem 2.6.4] ensures that, one can define a Hilbert space $\mathcal{H}_0^{\otimes k}$ and a multilinear mapping

$$\begin{array}{ccc} \mathcal{H}_0^k & \to & \mathcal{H}_0^{\otimes k} \\ (x_1, \cdots, x_k) & \mapsto & x_1 \otimes \cdots \otimes x_k \end{array}$$

which satify the universal property that for every "suitable" multilinear form $f : \mathcal{H}_0^k \to \mathbb{C}$ to an arbitrary Hilbert space $\mathcal{G}_0$, there exists a unique $g \in \mathcal{L}_b(\mathcal{H}_0^{\otimes k}, \mathcal{G}_0)$ such that for all $(x_1, \cdots, x_k) \in \mathcal{H}_0^k$, $f(x_1, \cdots, x_k) = g(x_1 \otimes \cdots \otimes x_k)$. We say that $(\mathcal{H}_0^{\otimes k}, \otimes)$ is a *k-th order tensor power* of $\mathcal{H}_0$ and that $\otimes$ is a *tensor product.* The couple $(\mathcal{H}_0^{\otimes k}, \otimes)$ is not unique but all k-th order tensor powers of $\mathcal{H}_0$ can be identifies in the sense that if $(\mathcal{H}_0^{\tilde{\otimes} k}, \tilde{\otimes})$ is another k-th order tensor power of $\mathcal{H}_0$, then there exists a unitary operator $U : \mathcal{H}_0^{\otimes k} \to \mathcal{H}_0^{\tilde{\otimes} k}$ such that for all $x_1, \cdots, x_k \in \mathcal{H}_0$, $x_1 \tilde{\otimes} \cdots \tilde{\otimes} x_k = U(x_1 \otimes \cdots \otimes x_k)$.

A simple but useful example is the space $L^2([0,1]^k)$ which is a k-th order tensor product of $L^2([0,1])$ with the tensor product defined for all $f_1, \cdots, f_k \in L^2([0,1])$ by

$$(f_1 \otimes \cdots \otimes f_k) : (t_1, \cdots, t_k) \mapsto \prod_{j=1}^{k} f_j(t_j) .$$

Another example is that, since $\mathcal{H}_0$ is a real Hilbert space, the space $\mathcal{S}_2(\mathcal{H}_0)$ is a second order tensor power of $\mathcal{H}_0$ with the tensor product defined for all $x_1, x_2 \in \mathcal{H}_0$ by $(x_1 \otimes x_2) = x_1 x_2^{\mathsf{H}}$ which is defined for all $u \in \mathcal{H}_0$ by $x_1 x_2^{\mathsf{H}} u = \langle u, x_2 \rangle_{\mathcal{H}_0} x_1$.

We now state the proposition defining the cumulant of k $\mathcal{H}_0$-valued random variables as an element of $\mathcal{H}_0^{\otimes k}$ (see [62, Proposition 3.12.6]).

Proposition 4.1. *Let $Y_1, \cdots, Y_k \in L^k(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, then there exists a unique vector of $\mathcal{H}_0^{\otimes k}$, denoted by $\mathrm{cum}(Y_1, \cdots, Y_k)$, such that for all $y_1, \cdots, y_k \in \mathcal{H}_0$,*

$$\langle \mathrm{cum}(Y_1, \cdots, Y_k), y_1 \otimes \cdots \otimes y_k \rangle_{\mathcal{H}_0^{\otimes k}} = \mathrm{cum}\left(\langle Y_1, y_1 \rangle_{\mathcal{H}_0}, \cdots, \langle Y_k, y_k \rangle_{\mathcal{H}_0} \right) .$$

When $k = 2$ and $\mathcal{H}_0$ is a real Hilbert space, taking $\mathcal{H}_0^{\otimes 2} = \mathcal{S}_2(\mathcal{H}_0)$ we see that, by definition, $\mathrm{cum}(Y_1, Y_2) = \mathrm{Cov}\,(Y_1, Y_2)$. When $\mathcal{H}_0 = L^2([0,1])$, taking $\mathcal{H}_0^{\otimes k} = L^2([0,1]^k)$, then for almost every $(t_1, \cdots, t_k) \in [0,1]^k$, we have

$$\mathrm{cum}(Y_1, \cdots, Y_k)(t_1, \cdots, t_k) \mapsto \mathrm{cum}(Y_1(t_1), \cdots, Y_k(t_k)) .$$

Since the notion of cumulants is a generalization of the covariance operator, the notion of stationarity can also be extended to orders larger than 2 (note that, with this definition, second-order stationarity is the same as weak stationarity).

Definition 4.2. *A sequence of $\mathcal{H}_0$-valued random variables $(X_t)_{t\in\mathbb{Z}}$ is said to be k-th order stationary if for all $t \in \mathbb{Z}$, $X_t \in L^k(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and for all $\ell \leq k$ and $t_1, \cdots, t_\ell \in \mathbb{Z}$, $\mathrm{cum}(X_{t_1+s}, \cdots, X_{t_\ell+s})$ does not depend on $s \in \mathbb{Z}$.*

For $\ell \in \mathbb{N}$ and $k \in \mathbb{N}^*$, we define the following assumption used in [62] and generalizing [8, Assumption 2.6.2].

Assumption 4.2 (ℓ, k)**.** *The time series $(X_t)_{t\in\mathbb{Z}}$ is k-th order stationary and*

$$\sum_{t_1,\cdots,t_{k-1}\in\mathbb{Z}} (1 + |t_j|^\ell) \left\| \mathrm{cum}(X_{t_1}, \cdots, X_{t_{k-1}}, X_0) \right\|_{\mathcal{H}_0^{\otimes k}} < +\infty \,. \tag{4.3}$$

4.1.4 Comparison of the assumptions

Assumptions based on a SCR may be seen as a restrictive (as, in particular they impose strong stationarity) but many usual models can be constructed in this way and [28, Section 2] gathers conditions for such models to be L^p-m-approximable. According to [15], Assumption 4.1(p) is weaker than L^p-m-approximability. Finally, the mixing condition and Assumption 4.2(ℓ, k) do not rely on any model but can be harder to verify in practice. However, a multitude of results have been derived under cumulants conditions for the finite dimensional case in [8, Sections 4, 5] thus giving a solid background to generalize to the infinite dimensional case. The author of [15] also provides a generalization of Assumption 4.1(p) to larger order dynamics and its link with Assumption 4.2(k, p).

4.2 An overview of recent advances

Now that we have recalled the main assumptions used in the literature, let us present the results obtained from them. We focus on consistency and asymptotic normality of estimators and note that we do not list all the assumptions needed to obtain the results. In particular, for results on the estimation of the eigenstructure of an operator, additional conditions ensuring idenfiability of the eigenvalues and eigenvectors are usually needed. For estimators involving a kernel or a window, assumptions on the bandwith are also made.

4.2.1 Inference for the covariance operator function

The covariance operator function is estimated by the *empirical covariance operator function* which we will denote by $(\hat{\Gamma}_X(h))_{h\in\mathbb{Z}}$. Its consistency is provided in [29] under L^4-m-approximability as a generalization of [30, Thm.16.1] which shows consistency of $\hat{\Gamma}_X(0)$ under the same assumption. Asymptotic normality of $\Gamma_X(0)$ is found in [39] under the same assumption. The eigenstructure of $\Gamma_X(0)$ playing a major role in Functional Principal Component Analysis (FPCA), its empirical counterpart has also been studied. For L^4-m-approximable time series, the eigenvalues and eigenvectors of $\hat{\Gamma}_X(0)$ are shown to be consistent estimators of the eigenvalues and eigenvectors of $\Gamma_X(0)$ in [30, Thm.16.2] and their asymptotically normal is proven in [39].

4.2.2 Asymptotic theory

The central limit theorem (CLT) for functional time series is obtained

- under α-mixing condition in [48] and other similar conditions in [47],
- under L^2-m-approximability in [31, 37] and
- under strict stationarity and Assumption 4.2($0, k$) for all $k \geq 2$ in [62, Cor.3.3.6].

In the two latter cases, the asymptotic covariance operator is the *long run covariance* $\sum_{h\in\mathbb{Z}} \Gamma_X(h)$ where the series converges in $\mathcal{S}_2(\mathcal{H}_0)$. An estimator of the long run covariance obtained by smoothing the empirical autocovariance operator function is provided in [31, 37] and

- its consistency is shown in [31] under L^2-m-approximability;
- its asymptotic normality is shown in [5] under assumptions closely related to L^4-m-approximability.

To reduce the dimension of the data, [5] proposes to consider the eigenstructure of the long run covariance instead of $\Gamma_X(0)$. This has the advantage of taking into accout covariance for lags greater tan 0. The eigenvalues and eigenvectors of the estimator of the long run covariance are shown to be asymptoticaly normal under assumptions closely related to L^4-m-approximability.

4.2.3 Estimation of the spectral density operator

In this section, we are interested in estimating (when it exists) the spectral density operator of a time series $(X_t)_{t\in\mathbb{Z}}$, *i.e.* the density f_X of its spectral operator measure ν_X with respect to Lebesgue's measure on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$. As in the univariate case, we want to express the density as the sum of the Fourier series of the covariance operator function. Before estimating it, let us recall conditions for such a series to converge and for its sum to be the density of ν_X with respect to Lebesgue's measure. For notation convenience, we write $\mathcal{S}_\infty(\mathcal{H}_0) := \mathcal{K}(\mathcal{H}_0)$ and $\|\cdot\|_\infty := \|\cdot\|$, and we define the following assumption for $1 \leq p \leq +\infty$.

Assumption 4.3 (p)**.** *The $\mathcal{H}_0$-valued time series $(X_t)_{t\in\mathbb{Z}}$ is weakly stationary and its covariance operator function Γ_X satisfies*

$$\sum_{h\in\mathbb{Z}} \|\Gamma_X(h)\|_p < +\infty \tag{4.4}$$

It is straightforward to see that, if $p \leq q$, then Assumption 4.3(p) implies Assumption 4.3(q) and that Assumption 4.3(2) is equivalent to Assumption 4.2(0, 2). Under such conditions, the Fourier series of the autocovariance operator function converges.

Proposition 4.2. *Suppose there exists $1 \leq p \leq +\infty$ such that $(X_t)_{t\in\mathbb{Z}}$ satisfies Assumption 4.3(p). Then the function*

$$f_X : \begin{array}{ccc} \mathbb{T} & \to & \mathcal{S}_p^+(\mathcal{H}_0) \\ \lambda & \mapsto & \sum_{h\in\mathbb{Z}} \Gamma_X(h)\mathrm{e}^{-\mathrm{i}\lambda h} \end{array} \tag{4.5}$$

is well-defined and continuous, which, in particular, gives $f_X \in \mathcal{L}^\infty(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathcal{S}_p(\mathcal{H}_0),\mathrm{Leb})$. Moreover, the following inversion formula holds, for all $h \in \mathbb{Z}$,

$$\Gamma_X(h) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda h} f_X(\lambda)\,\mathrm{d}\lambda\,. \tag{4.6}$$

The inversion formula (4.6) gives that, if ν_X is dominated by Lebesgue's measure on $\mathbb{T}$, then $\frac{\mathrm{d}\nu_X}{\mathrm{dLeb}} = f_X$ a.e. However, when $p \geq 2$, even though (4.6) holds, f_X is not necessarily the density of ν_X with respect to Lebesgue's measure in the sense defined in Theorem 2.4 since Proposition 4.2 does not necessarily imply $f_X \in \mathcal{L}^1(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathcal{S}_1(\mathcal{H}_0),\mathrm{Leb})$. Therefore, only Assumption 4.3(1) guarantees that ν_X is dominated by Lebesgue's measure on $\mathbb{T}$. This assumption (used in [50]) can be relaxed as follows (see [62, Proposition 2.3.5]).

Proposition 4.3. *Suppose there exists $1 \leq p \leq +\infty$ such that $(X_t)_{t\in\mathbb{Z}}$ satisfies Assumption 4.3(p) and*

$$\sum_{h\in\mathbb{Z}} |\mathrm{Tr}(\Gamma_X(h))| < +\infty\,. \tag{4.7}$$

Then f_X, defined in (4.5)*, satisfies $f_X \in \mathcal{L}^\infty(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathcal{S}_1(\mathcal{H}_0),\mathrm{Leb})$ and ν_X is dominated by Lebesgue's measure on $\mathbb{T}$ with density f_X.*

Even if Assumption 4.3(p) is not enough to have a proper spectral density operator (i.e. to have $\nu_X \ll \mathrm{Leb}$), we can still be interested in estimating f_X defined by (4.5). This explains why authors mainly focus on showing that their assumptions imply Assumption 4.3(p) for some $p \geq 2$ (usually $p = 2$) and not necessarily Equation (4.7). It is shown in [15] that Assumption 4.1(2) implies Assumption 4.3(2) and in [29] that L^2-m-approximability implies both Assumption 4.3(2) and Equation (4.7). For inference, two methods are commonly used. The first one (known as the *lag window estimator*) is based on smoothing the empirical covariance operator function in time while the second smoothes the periodogram in frequency. It is known that the two methods are equivalent if the weights used in the first case are the Fourier coefficients of the kernel used in the second (see [9, §10.4] or [15]). Note that, by estimating f_X, we also get an estimator of the long run covariance which is equal to $f_X(0)$.

Finally, since f_X is usually seen as a function from $\mathbb{T}$ to $\mathcal{S}_2(\mathcal{H}_0)$, consistency and asymptotic normality can have different forms. If $\hat{f}_X$ is an estimator of f_X one can ask for consistency for a given sequence of frequencies, uniformly in $\mathbb{T}$, or for some L^q norm. For asymptotic normality, one can show that $(\hat{f}_X(\lambda_j))_{j\in\{1,\cdots,d\}}$ is asymptoticaly normal as an element of $\mathcal{S}_2(\mathcal{H}_0)^d$ for given frequencies $\lambda_1,\cdots,\lambda_d$ or the stronger asymptotic normality of $\hat{f}_X$ as an element of $L^q(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathcal{S}_2(\mathcal{H}_0),\mathrm{Leb})$ for some $q\in\mathbb{N}^*\cup\{+\infty\}$. The same applies to the estimation of the eigenstructure of f_X.

The lag window estimator : The lag window estimator with a Bartlett window is studied in [29] where the authors show its integrated mean square consistency as well as the consistency of its eigenvalues and eigenvectors and the estimated dynamic functional principal component scores under L^4-m-approximability. Recently, the general lag window estimator (without restriction to the Barlett window) has also been studied.

- Its integrated mean square consistency is shown in [38] under L^4-m-approximability and uniform consistency is shown in [15] under Assumption 4.1(4).
- Its asymptotic normality as an element of $L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\mathcal{S}_2(\mathcal{H}_0),\mathrm{Leb})$ is shown for certain types of linear processes in [38]. Its asymptotic normality at given fixed frequencies is shown, in [15] under Assumption 4.1(4).
- Consistency of its eigenvalues and eigenprojectors are shown at fixed frequencies under Assumption 4.1(4) in [15].

Smoothing the periodogram : Assuming cumulant conditions, [62, Chapter 3] generalizes the results of [8, Sections 4, 5] to the infinite dimensional case. In particular, the author provides an estimator of f_X by smoothing the periodogram in frequency. Under Assumption 4.2(1, 2) and Assumption 4.2(1, 4) the author proves

- the uniform and integrated mean square consistency of the estimator of f_X;
- the uniform and integrated mean square consistency of its eigenvalues and
- the consistency of its eigenprojectors at a fixed frequency.

Under more restrictive assumptions (in particular Assumption 4.2$(0,k)$ for all $k\geq 2$), asymptotic normality of the estimator of f_X, its eigenvalues and eigenprojectors are also proven at given fixed frequencies.

4.3 A note on discrete observations

When the space $\mathcal{H}_0$ is a functional space (*e.g.* $L^2([0,1])$), the observation X_t is seen as a function $u\mapsto X_t(u)$. For practical applications, such a function is usually decomposed in a given basis of functions (*e.g.* Fourier basis or splines) but one can also consider the discrete observation framework where we only have access to measurements of the type

$$X_t(u_{t,j})+\epsilon_{t,j}\ ,\quad t=0,\cdots,T-1,\ j=1\cdots,N_t$$

where $(\epsilon_{t,j})_{t=0,\cdots T-1,j=1,\cdots N_t}$ is a noise. This is the basis of *sparse functiondal data* which has been studied in the context of independent data (see *e.g.* [65, 26, 42]). For dependent data, the effect of discrete observations has not been sudied until recently in [62, Section 3.8] and [56]. In this scenario, estimators have to be adapted and non-parametric methods are used.

5 Postponed proofs

5.1 Proofs of Section 2.1 and Section 2.2

Proof of Lemma 2.1. By (2.3), we only need to show that, if Φ is simply measurable then it is measurable. The space $\mathcal{E}$ is separable because the set of finite rank operators is dense in $\mathcal{E}$ for the norm $\|\cdot\|$ if $\mathcal{E}=\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$ and $\|\cdot\|_p$ if $\mathcal{E}=\mathcal{S}_p(\mathcal{H}_0,\mathcal{G}_0)$. By Pettis's measurability theorem (see [17, Theorem II.1.2]), this implies that it is enough to show that for all $f\in\mathcal{E}^*$, $f\circ\Phi$ is a measurable complex-valued function. By [13, Theorems 19.1, 18.14, 19.2] we get that $\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)^*$, $\mathcal{S}_1(\mathcal{H}_0,\mathcal{G}_0)^*$ and $\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0)^*$ are respectively isometrically isomorphic to

$\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)$, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and the duality relation can be defined on $\mathcal{E} \times \mathcal{E}^*$ as $(\mathrm{P}, \mathrm{Q}) \mapsto \mathrm{Tr}(\mathrm{Q}^{\mathsf{H}}\mathrm{P})$. This means that we only have to show measurability of the complex-valued functions $\lambda \mapsto \mathrm{Tr}(\mathrm{P}^{\mathsf{H}}\Phi(\lambda))$ for all $\mathrm{P} \in \mathcal{E}^*$. Let $(\phi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}$ be Hilbert basis of $\mathcal{H}_0$ and $\mathcal{G}_0$ respectively, then $\mathrm{Tr}(\mathrm{P}^{\mathsf{H}}\Phi(\lambda)) = \sum_{k\in\mathbb{N}} \langle \Phi(\lambda)\phi_k, \mathrm{P}\psi_k \rangle_{\mathcal{G}_0}$ which defines a measurable function of λ by simple measurability of Φ. □

Proof of Lemma 2.3. The first point comes from the fact that for all $A \in \mathcal{A}$, $\nu(A) \preceq \nu(\Lambda)$. Now, if ν is trace-class, then (2.4) is easily verified for the norm $\|\cdot\|_1$ using the fact that $\|\cdot\|_1 = \mathrm{Tr}(\cdot)$ for positive operators. Finally, by definition of $\|\nu\|_1$, regularity of $\|\nu\|_1$ is equivalent to regularity of ν as a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which clearly implies regularity of $y^{\mathsf{H}}\nu x$ for all $x, y \in \mathcal{H}_0$. Suppose now that for all $x, y \in \mathcal{H}_0$, $y^{\mathsf{H}}\nu x$ is regular, then let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert-basis of $\mathcal{H}_0$, and define for all $n \in \mathbb{N}$, the non-negative measure $\mu_n := \sum_{k=0}^{n} e_k^{\mathsf{H}}\nu e_k$ such that for all $A \in \mathcal{A}$, $\|\nu\|_1(A) = \lim_{n\to+\infty} \mu_n(A) = \sup_{n\in\mathbb{N}} \mu_n(A)$. Then, by Vitali-Hahn-Sakh-Nikodym's theorem (see [11]), the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly countably additive which implies regularity of $\|\nu\|_1$ by [17, Lemma VI.2.13]. □

Proof of Theorem 2.4. Suppose $\|\nu\|_1 \ll \mu$, then, since $\mathcal{S}_1(\mathcal{H}_0)$ is separable and is the dual of $\mathcal{K}(\mathcal{H}_0)$, it is a *separable dual space* and [17, theorem III.3.1] gives the existence and uniqueness of a density $g \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ satisfying (2.5). Then for all $x \in \mathcal{H}_0$, for all $A \in \mathcal{A}$,

$$\int_A \langle g(\lambda)x, x \rangle_{\mathcal{H}_0} \, \mu(\mathrm{d}\lambda) = \langle \nu(A)x, x \rangle_{\mathcal{H}_0} \geq 0$$

and therefore there exists a set $A_x \in \mathcal{A}$ with $\mu(A_x) = 0$ and $\langle g(\lambda)x, x \rangle_{\mathcal{H}_0} \geq 0$ for all $\lambda \in A_x^c$. Taking $(x_n)_{n\in\mathbb{N}}$ a dense countable subset of $\mathcal{H}_0$ we get that g is positive on $\left(\bigcup_{n\in\mathbb{N}} A_{x_n}\right)^c$ where $\mu\left(\bigcup_{n\in\mathbb{N}} A_{x_n}\right) = 0$ thus proving Assertion (a). Moreover, taking the trace in (2.5) gives for all $A \in \mathcal{A}$,

$$\|\nu\|_1(A) = \int_A \|g\|_1 \, \mathrm{d}\mu$$

which gives Assertion (b) and implies easily Assertion (c). The converse implication is a consequence of Assertion (b). □

5.2 Proofs of Section 2.5 and Section 2.6

Proof of Proposition 2.7. The proof is easily derived from the fact that $g = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1} \|g\|_1$ (see Theorem 2.4) and the definition of $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Note that $\Phi g \Phi^{\mathsf{H}} \in \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0))$ and $\Phi g^{1/2} \in \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0))$ by simple-measurability of Φ and g and Lemma 2.1. □

Proof of Proposition 2.8. All theses results, except Relation (2.12), are easily derived from the characterization of Proposition 2.7 and the module nature of $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$. We now prove Relation (2.12). First note that $\|\nu\|_1(\{g = 0\}) = \int_{\{g=0\}} \|g\|_1 \, \mathrm{d}\mu = 0$ and therefore

$$\begin{aligned}
\|\nu\|_1 \left(\{\Phi f^{1/2} \neq 0\}\right) &= \|\nu\|_1 \left(\{\Phi f^{1/2} \neq 0\} \cap \{g \neq 0\}\right) \\
&= \|\nu\|_1 \left(\left\{\frac{\Phi f^{1/2}}{\|g\|_1^{1/2}} \neq 0\right\} \cap \{g \neq 0\}\right) \\
&= \|\nu\|_1 \left(\{\Phi g^{1/2} \neq 0\} \cap \{g \neq 0\}\right) \\
&\leq \|\nu\|_1 \left(\{\Phi g^{1/2} \neq 0\}\right)
\end{aligned}$$

which gives inclusion ($\subset$) of (2.12) since $\|\nu\|_1 \ll \mu$. Conversely, if $\|\nu\|_1 \left(\{\Phi f^{1/2} \neq 0\}\right) = 0$, then

$$\mu\left(\{\Phi g^{1/2} \neq 0\}\right) = \mu\left(\{\Phi f^{1/2} \|g\|_1^{1/2} \neq 0\}\right) = \mu\left(\{\Phi f^{1/2} \neq 0\} \cap \{g \neq 0\}\right) = 0$$

because $0 = \|\nu\|_1 \left(\{\Phi f^{1/2} \neq 0\}\right) = \int_{\{\Phi f^{1/2} \neq 0\}} \|g\|_1 \, \mathrm{d}\mu$. □

Proof of Proposition 2.9. Since $\|\nu\|_1(\{g=0\})=0$ and $g=f\|g\|_1$, where $f=\frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$, we get

$$\begin{aligned}\|\nu\|_1\left(\left\{\operatorname{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)&=\|\nu\|_1\left(\left\{\operatorname{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq 0\}\right)\\&=\|\nu\|_1\left(\left\{\operatorname{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq 0\}\right)\\&\leq\|\nu\|_1\left(\left\{\operatorname{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)\end{aligned}$$

which gives (i') $\Rightarrow$ (i) since $\|\nu\|_1\ll\mu$. Conversely, if $\|\nu\|_1\left(\left\{\operatorname{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)=0$, then

$$\mu\left(\left\{\operatorname{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)=\mu\left(\left\{\operatorname{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq 0\}\right)=0$$

because $0=\|\nu\|_1\left(\left\{\operatorname{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)=\int_{\left\{\operatorname{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}}\|g\|_1\,\mathrm{d}\mu$. Hence (i') $\Leftrightarrow$ (i).

Moreover, Equivalences (ii) $\Leftrightarrow$ (ii') and (iii) $\Leftrightarrow$ (iii') and Relation (2.14) are easy consequences of the fact that $g=f\|g\|_1$ and the other results come easily using the definition of $\mathscr{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$. Again, note that measurability of $\Phi g^{1/2}$ and $(\Phi g^{1/2})(\Phi g^{1/2})$ are ensured by $\mathcal{O}$-measurability of Φ, simple measurability of f and Lemma 2.1. $\square$

Proof of Theorem 2.10. As for Proposition 2.8, these results come easily using the definition and Identity (2.16). Relation (2.18) is proven the same way as (2.12). $\square$

Proof of Theorem 2.11. In the first two steps of the proof of [33, Theorem 3.4.12], [45, Theorem 4.22] the authors show that, if $\Phi\in\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$ and $\epsilon>0$, there exists $\Psi\in L^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\|\nu\|_1)\subset\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$ such that $\|\Phi-\Psi\|_{\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)}<\epsilon$. This implies that $L^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\|\nu\|_1)$ is dense in $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$. The other results follow using (2.20) and density of simple functions and trigonometric polynomials in $L^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\|\nu\|_1)$ (see Theorem C.1). $\square$

5.3 Proofs of Section 2.7

The proof of Theorem 2.12 relies on the following lemma.

Lemma 5.1. *Let $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ be separable Hilbert spaces and $\mathrm{P}\in\mathcal{O}(\mathcal{G}_0,\mathcal{I}_0)$, $\mathrm{Q}\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$. The following assertions hold.*

(i) $\operatorname{Im}(|\mathrm{Q}^{\mathsf{H}}|)=\operatorname{Im}(\mathrm{Q})$.

(ii) If $\operatorname{Im}(\mathrm{Q})\subset\mathcal{D}(\mathrm{P})$, *then* $(\mathrm{PQ})(\mathrm{PQ})^{\mathsf{H}}=(\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|)(\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|)^{\mathsf{H}}$.

(iii) If $\operatorname{Im}(\mathrm{Q})\subset\mathcal{D}(\mathrm{P})$, *then* $\mathrm{PQ}\in\mathcal{S}_2(\mathcal{H}_0,\mathcal{I}_0)$ *if and only if* $\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|\in\mathcal{S}_2(\mathcal{G}_0,\mathcal{I}_0)$. *In this case* $\|\mathrm{PQ}\|_2=\|\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|\|_2$.

Proof. Let us consider the singular values decompositions

$$\mathrm{Q}=\sum_{n\in\mathbb{N}}\sigma_n\psi_n\otimes\phi_n\quad\text{and}\quad\left|\mathrm{Q}^{\mathsf{H}}\right|=\sum_{n\in\mathbb{N}}\sigma_n\psi_n\otimes\psi_n\;.$$

Proof of (i). We have $\operatorname{Im}(\mathrm{Q})=\left\{\sum_{n\in\mathbb{N}}\sigma_n x_n\psi_n\;:\;(x_n)_{n\in\mathbb{N}}\in\ell^2(\mathbb{N})\right\}=\operatorname{Im}(|\mathrm{Q}^{\mathsf{H}}|)$.
Proof of (ii). By the first point both composition PQ and $\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|$ make sense. Consider the polar decomposition of Q^{H} : $\mathrm{Q}^{\mathsf{H}}=U\,|\mathrm{Q}^{\mathsf{H}}|$, then $\mathrm{Q}=|\mathrm{Q}^{\mathsf{H}}|\,U^{\mathsf{H}}$ and we get

$$(\mathrm{PQ})(\mathrm{PQ})^{\mathsf{H}}=\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)U^{\mathsf{H}}U\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)^{\mathsf{H}}=\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)P_{\ker(|\mathrm{Q}^{\mathsf{H}}|)^\perp}\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)^{\mathsf{H}}=\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)^{\mathsf{H}}$$

because $|\mathrm{Q}^{\mathsf{H}}|\,P_{\ker(|\mathrm{Q}^{\mathsf{H}}|)^\perp}=|\mathrm{Q}^{\mathsf{H}}|$.
Proof of (iii). We have that $\mathrm{PQ}\in\mathcal{S}_2(\mathcal{H}_0,\mathcal{I}_0)$ if and only if $(\mathrm{PQ})(\mathrm{PQ})^{\mathsf{H}}\in\mathcal{S}_1(\mathcal{I}_0)$, which is equivalent to $\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|\in\mathcal{S}_2(\mathcal{G}_0,\mathcal{I}_0)$ by the previous point. $\square$

Proof of Theorem 2.12. Let μ be a dominating measure for $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then, by definition of $\Phi\nu\Phi^{\mathsf{H}}$, μ also dominates $\left\|\Phi\nu\Phi^{\mathsf{H}}\right\|_1$ and $\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu} = (\Phi g^{1/2})(\Phi g^{1/2})^{\mathsf{H}}$. Hence, $\left(\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu}\right)^{1/2} = \left|(\Phi g^{1/2})^{\mathsf{H}}\right|$ and we get, by Proposition 2.9,

$$\begin{aligned}
\Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) &\Leftrightarrow \begin{cases} \mathrm{Im}\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \subset \mathcal{D}(\Psi) \ \ \mu\text{-a.e.} \\ \Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0), \mu) \end{cases} \\
&\Leftrightarrow \begin{cases} \mathrm{Im} g^{1/2} \subset \mathcal{D}(\Psi\Phi) \ \ \mu\text{-a.e.} \\ \Psi\Phi g^{1/2} \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0), \mu) \end{cases} \\
&\Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)
\end{aligned}$$

where the second equivalence comes from Lemma 5.1 and the fact that for all $\lambda \in \Lambda$, $\mathcal{D}(\Psi(\lambda)\Phi(\lambda)) = \Phi(\lambda)^{-1}(\mathcal{D}(\Psi(\lambda)))$ which gives that $\mathrm{Im}(g^{1/2}(\lambda)) \subset \mathcal{D}(\Psi(\lambda)\Phi(\lambda))$ if and only if $\mathrm{Im}(\Phi(\lambda)g^{1/2}(\lambda)) \subset \mathcal{D}(\Psi(\lambda))$. Moreover, Assertion (a) holds because for all $\Psi, \Theta \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ and $A \in \mathcal{A}$,

$$\begin{aligned}
(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}}(A) &= \int_A \left(\Psi\Phi g^{1/2}\right)\left(\Theta\Phi g^{1/2}\right)^{\mathsf{H}} \mathrm{d}\mu \\
&= \int_A \left(\Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)\left(\Theta\left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)^{\mathsf{H}} \mathrm{d}\mu \qquad \text{(by lemma 5.1)} \\
&= \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}}(A)
\end{aligned}$$

which also gives Assertion (b) by taking $A = \Lambda$. Finally, to show Assertion (c), suppose that Φ is injective $\|\nu\|_1$-a.e. then $\Phi^{-1}\Phi : \lambda \mapsto \mathrm{Id}_{\mathcal{H}_0}\mathbb{1}_{\{\Phi(\lambda) \text{ is injective}\}}$ is in $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$ which gives that $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}})$ by Assertion (a). □

5.4 Proofs of Section 2.8

Proof of Theorem 2.13. We set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. For all $A, B \in \mathcal{A}$ and $\mathrm{P}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, we have, by Theorem 2.10,

$$\begin{aligned}
[\mathbb{1}_A\mathrm{P}, \mathbb{1}_B\mathrm{Q}]_{\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_W)} &= \mathrm{P}\nu_W(A \cap B)\mathrm{Q}^{\mathsf{H}} \\
&= \mathrm{P}\,\mathrm{Cov}\,(W(A), W(B))\,\mathrm{Q}^{\mathsf{H}} \\
&= \mathrm{Cov}\,(\mathrm{P}W(A), \mathrm{Q}W(B)) \\
&= [\mathrm{P}W(A), \mathrm{Q}W(B)]_{\mathcal{G}} .
\end{aligned}$$

Then Proposition 2.5, applied to $J = \mathcal{A} \times \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ with $v_{(A,\mathrm{P})} = \mathbb{1}_A\mathrm{P}$ and $w_{(A,\mathrm{P})} = \mathrm{P}W(A)$, gives that there exists a unique gramian-isometry

$$I_W^{\mathcal{G}_0} : \overline{\mathrm{Span}}^{\widehat{\mathcal{H}}^{\nu_W,\mathcal{G}_0}}(\mathbb{1}_A\mathrm{QP} \ : \ A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)) \to \mathcal{G}$$

such that for all $A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, $I_W^{\mathcal{G}}(\mathbb{1}_A\mathrm{P}) = \mathrm{P}W(A)$ and, in addition,

$$\mathrm{Im}(I_W^{\mathcal{G}_0}) = \overline{\mathrm{Span}}^{\mathcal{G}}(\mathrm{QP}W(A) \ : \ A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)) . \tag{5.1}$$

Now, note that

$$\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) = \{\mathrm{QP} \ : \ \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)\} . \tag{5.2}$$

This gives

$$\mathrm{Span}\,(\mathbb{1}_A\mathrm{QP} \ : \ A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)) = \mathrm{Span}\,(\mathbb{1}_A\mathrm{P} \ : \ A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$$

and therefore, by Theorem 2.11,

$$\overline{\mathrm{Span}}^{\widehat{\mathcal{H}}^{\nu_W,\mathcal{G}_0}}(\mathbb{1}_A\mathrm{QP} \ : \ A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)) = \widehat{\mathcal{H}}^{\nu_W,\mathcal{G}_0} .$$

Finally, (5.2) with (5.1) gives

$$\mathrm{Im}(I_W^{\mathcal{G}_0}) = \overline{\mathrm{Span}}^{\mathcal{G}}(\mathrm{P}W(A) \ : \ A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)) = \mathcal{H}^{W,\mathcal{G}_0}$$

which concludes the proof. □

Proof of Theorem 2.14. Let $(\phi_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathcal{H}_0$, then for all $A \in \mathcal{A}$,

$$\mathbb{E}\left[\sum_{n\in\mathbb{N}} \left|w(\mathbb{1}_A\phi_n^{\mathsf{H}})\right|^2\right] = \sum_{n\in\mathbb{N}} \left\|w(\mathbb{1}_A\phi_n^{\mathsf{H}})\right\|^2_{\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})} = \sum_{n\in\mathbb{N}} \left\|\mathbb{1}_A\phi_n^{\mathsf{H}}\right\|^2_{\widehat{\mathcal{H}}^{\nu,\mathbb{C}}} = \sum_{n\in\mathbb{N}} \phi_n^{\mathsf{H}}\nu(A)\phi_n$$
$$= \mathrm{Tr}(\nu(A)) < +\infty \quad (5.3)$$

Hence

$$\sum_{n\in\mathbb{N}} \left|w(\mathbb{1}_A\phi_n^{\mathsf{H}})\right|^2 < +\infty \quad \mathbb{P}\text{-}a.s.$$

We can thus define for all $A \in \mathcal{A}$, a random variable $W(A)$ such that

$$W(A) \overset{\mathbb{P}\text{-}a.s}{=} \sum_{n\in\mathbb{N}} w(\mathbb{1}_A\phi_n^{\mathsf{H}})\phi_n \; .$$

We then have $W(A) \in \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ by (5.3) and since $\mathbb{E}\left[w(\mathbb{1}_A\phi_n^{\mathsf{H}})\right] = 0$ for all $n \in \mathbb{N}$. Furthermore, by linearity and continuity of w, we have, for all $A \in \mathcal{H}_0$, for all $x \in \mathcal{H}_0$

$$\begin{aligned}
\langle W(A), x\rangle_{\mathcal{H}_0} &\overset{\mathbb{P}\text{-}a.s}{=} \sum_{n\in\mathbb{N}} w(\mathbb{1}_A\phi_n^{\mathsf{H}})\, \langle \phi_n, x\rangle_{\mathcal{H}_0} \\
&= w\left(\mathbb{1}_A \sum_{n\in\mathbb{N}} \langle x, \phi_n\rangle_{\mathcal{H}_0} \phi_n^{\mathsf{H}}\right) \\
&= w(\mathbb{1}_A x^{\mathsf{H}}) \qquad (5.4)
\end{aligned}$$

It follows that for all $A, B \in \mathcal{A}$, and $x, y \in \mathcal{H}_0$,

$$\begin{aligned}
y^{\mathsf{H}}\,\mathrm{Cov}\,(W(A), W(B))\, x &= \mathrm{Cov}\,(\langle W(A), y\rangle\, , \langle W(B), x\rangle) \\
&= \left\langle w(\mathbb{1}_A y^{\mathsf{H}}), w(\mathbb{1}_B x^{\mathsf{H}})\right\rangle_{\mathcal{M}(\Omega,\mathcal{F},\mathbb{P})} \\
&= \left\langle \mathbb{1}_A y^{\mathsf{H}}, \mathbb{1}_B x^{\mathsf{H}}\right\rangle_{\widehat{\mathcal{H}}^{\nu,\mathbb{C}}} \qquad \text{(because } w \text{ is an isometry)} \\
&= y^{\mathsf{H}}\nu(A\cap B)x \; .
\end{aligned}$$

Hence $\mathrm{Cov}\,(W(A), W(B)) = \nu(A\cap B)$ which implies that W is an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν. Finally, (5.4) and Theorem 2.13 give that $w = I_W^{\mathbb{C}}$ and therefore (2.25) holds.

Let us now suppose that another $\mathcal{H}_0$-valued random c.a.g.o.s. measure V on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ satisfies (2.25), then for $\varphi = \mathbb{1}_A\phi_n^{\mathsf{H}}$, we get for all $A \in \mathcal{A}$ and $n \in \mathbb{N}$,

$$\langle V(A), \phi_n\rangle_{\mathcal{H}_0} = w(\mathbb{1}_A\phi_n^{\mathsf{H}}) \quad \mathbb{P}\text{-a.s.}$$

With (5.4), we get that, for all $n \in \mathbb{N}$, $\langle V(A), \phi_n\rangle_{\mathcal{H}_0} = \langle W(A), \phi_n\rangle_{\mathcal{H}_0}$ $\mathbb{P}$-a.s. and therefore for all $A \in \mathcal{A}$, $W(A) = V(A)$ $\mathbb{P}$-a.s.

□

5.5 Proofs of Section 2.9

Proof of Corollary 2.16.
Proof of Assertion (i). This follows from Assertion (b) of Theorem 2.12 and Theorem 2.13.
Proof of Assertion (ii). If $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$, then the equivalence between $W \in \hat{\mathcal{S}}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$ is just another formulation of the equivalence (2.21). The identity (2.28) is equivalent to show that, for all $\Psi \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu_V)$ and $A \in \mathcal{A}$,

$$\int_A \Psi \,\mathrm{d}V = \int_A \Psi\Phi \,\mathrm{d}W, \qquad (5.5)$$

where we set $V = \hat{F}_\Phi(W)$. We first prove (5.5) for Ψ of the form $\Psi = \mathrm{P}\mathbb{1}_B$ with $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$ and $B \in \mathcal{A}$. Calling $C = A \cap B$, (5.5) becomes

$$\mathrm{P}\left(\int_C \Phi(\lambda)\, W(\mathrm{d}\lambda)\right) = \int_C \mathrm{P}\Phi(\lambda)\, W(\mathrm{d}\lambda) \; . \qquad (5.6)$$

When $\mathcal{I}_0 = \mathcal{G}_0$ this identity comes from the fact that the integral with respect to W is $\mathcal{L}_b(\mathcal{G}_0)$-linear. When $\mathcal{I}_0 \neq \mathcal{G}_0$, we have to show it by hand. Using the notations $I_W^{\mathcal{G}_0}$ and $I_W^{\mathcal{I}_0}$, Relation (5.6) is equivalent to

$$I_W^{\mathcal{I}_0}(\mathbb{1}_C \mathrm{P}\Phi) = \mathrm{P} I_W^{\mathcal{G}_0}(\mathbb{1}_C \Phi) \tag{5.7}$$

If $\Phi = \mathrm{Q}\mathbb{1}_D$ with $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $D \in \mathcal{A}$, then $\mathrm{PQ} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{I}_0)$ and we immediately get

$$I_W^{\mathcal{I}_0}(\mathrm{P}\Phi) = I_W^{\mathcal{I}_0}(\mathrm{PQ}\mathbb{1}_D) = \mathrm{PQ}W(D) = \mathrm{P} I_W^{\mathcal{G}_0}(\mathrm{Q}\mathbb{1}_D) = \mathrm{P} I_W^{\mathcal{G}_0}(\Phi)$$

This property extends to the case Φ is a simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued function by linearity and if $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, Theorem 2.11 gives that there exists a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions converging to Φ in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Hence, calling $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$, we get

$$\begin{aligned}
\|\mathrm{P}\Phi - \mathrm{P}\Phi_n\|^2_{\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{I}_0),\nu)} &= \int \left\|\mathrm{P}(\Phi - \Phi_n) f^{1/2}\right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&\leq \|\mathrm{P}\| \int \left\|(\Phi - \Phi_n) f^{1/2}\right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \|\mathrm{P}\| \|\Phi - \Phi_n\|^2_{\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)} \\
&\xrightarrow[n\to+\infty]{} 0 \; .
\end{aligned}$$

Since for all $n \in \mathbb{N}$, $I_W^{\mathcal{I}_0}(\mathrm{P}\Phi_n) = I_W^{\mathcal{G}_0}(\Phi_n)$ and by continuity of $I_W^{\mathcal{I}_0}$ and $I_W^{\mathcal{G}_0}$, we finally get (5.7), that is (5.5) for $V = \hat{F}_\Phi(W)$ and $\Psi = \mathrm{P}\mathbb{1}_C$ with $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$ and $C \in \mathcal{A}$. By linearity, it follows that (5.5) holds with $V = \hat{F}_\Phi(W)$ and all simple $\mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$-valued function Ψ.

Finally, if $\Psi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu)$, then, by Theorem 2.11, there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of simple $\mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$-valued functions converging to Ψ in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu)$. Since $\Psi \mapsto \Psi\Phi$ is a gramian-isometry from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu_V)$ to $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu_W)$ (see Theorem 2.12), the sequence $(\Psi_n\Phi)_{n\in\mathbb{N}}$ then converges to $\Psi\Phi$ in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu_W)$ and by continuity of the stochastic integral we get

$$\int_A \Psi \mathrm{d}V = \lim_{n\to+\infty} \int_A \Psi_n \mathrm{d}V = \lim_{n\to+\infty} \int_A \Psi_n \Phi \mathrm{d}W = \int_A \Psi\Phi \mathrm{d}W \; .$$

Proof of Assertion (iii). Set $V = \hat{F}_\Phi(W)$ and denote by $\nu_V = \Phi\nu\Phi^{\mathsf{H}}$ the spectral operator measure of V. Supposing that Φ is injective $\|\nu_W\|_1$-a.e, Assertions (c) and (a) of Theorem 2.12, give that $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \nu_V)$ (*i.e.* $V \in \hat{\mathcal{S}}_{\Phi^{-1}}(\Omega, \mathcal{F}, \mathbb{P})$) and $\Phi^{-1}\nu_V \left(\Phi^{-1}\right)^{\mathsf{H}} = \nu_W$. Hence, writing Relation (2.28) with $\Psi = \Phi^{-1}$, we get $\hat{F}_{\Phi^{-1}}(V) = \hat{F}_{\Phi^{-1}\Phi}(W) = W$. Moreover, reversing the roles of W and V in assertion (i) gives the reciprocal $\supseteq_{\approx}$ in (2.27). □

5.6 Proofs of Section 3

Proof of Theorem 3.2. If X is weakly-stationary then, by Lemma 3.1, the family of shifts $(U_h^X)_{h\in\mathbb{G}}$ is a c.g.u.r. of $\mathbb{G}$ on $\mathcal{H}^X$. Hence Theorem 2.6 gives that there exists a unique regular gramian-projection valued measure ξ^X on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{H}^X)$ such that, for all $h \in \mathbb{G}$,

$$U_h^X = \int \chi(h)\, \xi^X(\mathrm{d}\chi) \; . \tag{5.8}$$

Then the mapping

$$\hat{X} : \begin{array}{ccc} \mathcal{B}(\hat{\mathbb{G}}) & \to & \mathcal{H}^X \\ A & \mapsto & \xi^X(A)X_0 \end{array}$$

is a c.a.g.o.s. measure on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{H}^X)$ and is regular because for all $Y \in \mathcal{H}^X$, $\left\langle \hat{X}(\cdot), Y \right\rangle_{\mathcal{H}^X} = Y^{\mathsf{H}}\xi^X(\cdot)X_0$ is regular. Since $\mathcal{H}^X$ is gramian isometrically embeded in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, $\hat{X}$ is also a regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \Omega, \mathcal{F}, \mathbb{P})$. Then, from (5.8) we get that for all $t \in \mathbb{G}$,

$$X_t = U_t^X X_0 = \int \chi(t)\, \xi^X(\mathrm{d}\chi) X_0 = \int \chi(t)\, \hat{X}(\mathrm{d}\chi) \; .$$

To show uniqueness, suppose there exists another regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure W on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \Omega, \mathcal{F}, \mathbb{P})$ satisfying (3.1). Then the gramian-isometries $I_{\hat{X}}^{\mathcal{H}_0}$ and $I_W^{\mathcal{H}_0}$ coincide on $\mathrm{Span}\,(\chi \mapsto \chi(t)\mathrm{P} \,:\, t \in \mathbb{G}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0))$ which implies, by Theorem 2.11, that they coincide on $\mathsf{L}^2(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{O}(\mathcal{H}_0), \nu_X) \cap \mathsf{L}^2(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{O}(\mathcal{H}_0), \nu_W)$. In particular for all $A \in \mathcal{A}$ we get

$$W(A) = I_W^{\mathcal{H}_0}(\mathbb{1}_A \mathrm{Id}_{\mathcal{H}_0}) = I_{\hat{X}}^{\mathcal{H}_0}(\mathbb{1}_A \mathrm{Id}_{\mathcal{H}_0}) = \hat{X}(A)$$

thus proving uniqueness of $\hat{X}$.

Relation (3.2) follows from (3.1) and the gramian-isometric property of $I_{\hat{X}}^{\mathcal{H}_0}$ and uniqueness of ν_X comes from (3.2) and Theorem C.1.

Finally, we show the converse statement in Theorem 3.2. Suppose that there exists a regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \Omega, \mathcal{F}, \mathbb{P})$ satisfying (3.1). Then, the first two points of Definition 1.3 are straightforward and, calling ν_X the intensity operator measure of $\hat{X}$ and using the gramian-isometric property of integration with respect to $\hat{X}$, we get for all $t, h \in \mathbb{G}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \int \chi(t+h)\overline{\chi(t)}\nu_X(\mathrm{d}\chi) = \int \chi(h)\nu_X(\mathrm{d}\chi)$ which gives the third point of Definition 1.3. Finally, for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, for all $h \in \mathbb{G}$,

$$\mathrm{Tr}(\mathrm{P}\Gamma(h)) = \mathrm{Tr}\left(\mathrm{P}\int \chi(h)\,\nu_X(\mathrm{d}\chi)\right) = \int \chi(h)\mathrm{Tr}(\mathrm{P}f_{\nu_X}(\chi))\,\|\nu_X\|_1(\mathrm{d}\chi)$$

and, since for $\|\nu_X\|_1$-almost every $\chi \in \hat{\mathbb{G}}$, $|\mathrm{Tr}(\mathrm{P}f_{\nu_X}(\chi))| \le \|\mathrm{P}\|\|f_{\nu_X}(\chi)\|_1 = \|\mathrm{P}\|$, we get continuity of $h \mapsto \mathrm{Tr}(\mathrm{P}\Gamma(h))$ by Lebesgue's dominated convergence theorem thus showing the last point of Definition 1.3. $\square$

6 Concluding remarks

6.1 Bochner's and Stone's theorems for normal Hilbert modules

In the following, we consider an l.c.a. group $(\mathbb{G}, +)$ and a Hilbert space $\mathcal{H}_0$. We discuss here the relations between Bochner's and Stone's theorem and their generalizations for the Hilbert valued case.

Definition 6.1 (Hermitian non-negative definite function)**.** *A function $\gamma : \mathbb{G} \to \mathbb{C}$ defined on an l.c.a. group $(\mathbb{G}, +)$ is said to be hermitian non-negative definite if for all $n \in \mathbb{N}$, $t_1, \cdots, t_n \in \mathbb{G}$ and $a_1, \cdots, a_n \in \mathbb{C}$,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \gamma(t_i - t_j) \ge 0.$$

Theorem 6.1 (Bochner)**.** *Let $\mathbb{G}$ be an l.c.a. group and $\gamma : \mathbb{G} \to \mathbb{C}$ be a continuous hermitian non-negative definite function. Then there exists a unique regular finite non-negative measure μ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}))$ such that*

$$\gamma(h) = \int \mathrm{e}_h \,\mathrm{d}\mu = \int \chi(h)\,\mu(\mathrm{d}\chi), \quad h \in \mathbb{G}. \tag{6.1}$$

Theorem 6.2 (Stone)**.** *Let $\mathbb{G}$ be an l.c.a. group and $U : \begin{array}{ccc} \mathbb{G} & \mapsto & \mathcal{L}_b(\mathcal{H}_0) \\ h & \mapsto & U_h \end{array}$ be a c.u.r. of $\mathbb{G}$ on a Hilbert space $\mathcal{H}_0$. Then there exists a unique regular projection-valued measure ξ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}))$ such that*

$$U_h = \int \mathrm{e}_h \,\mathrm{d}\xi = \int \chi(h)\,\xi(\mathrm{d}\chi), \quad h \in \mathbb{G}. \tag{6.2}$$

Stone's theorem can be seen as a generalization of Bochner's theorem for operator-valued functions. However, it is not necessary to restrict ourselves to unitary representations of $\mathbb{G}$ on $\mathcal{H}_0$ and, using an appropriate definition for hermitian non-negative definite operator-valued functions, one can show that Bochner's theorem still holds. We introduce the two following definitions which will be proved to be equivalent.

Definition 6.2 (Hermitian non-negative definite operator-valued function)**.** *Let* $(\mathbb{G}, +)$ *be an l.c.a. group and* $\mathcal{H}_0$ *a Hilbert space. Then a function* $\Gamma : \mathbb{G} \to \mathcal{L}_b(\mathcal{H}_0)$ *is said to be hermitian non-negative definite if for all* $n \in \mathbb{N}$*,* $t_1, \cdots, t_n \in \mathbb{G}$ *and* $a_1, \cdots, a_n \in \mathbb{C}$*,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \Gamma(t_i - t_j) \succeq 0.$$

Equivalently, Γ *is hermitian non-negative definite if and only if for all* $x \in \mathcal{H}_0$*,* $t \mapsto \langle \Gamma(t)x, x \rangle_{\mathcal{H}_0}$ *is hermitian non-negative definite.*

Definition 6.3 (Positive-type operator-valued function)**.** *Let* $(\mathbb{G}, +)$ *be an l.c.a. group and* $\mathcal{H}_0$ *a Hilbert space. Then a function* $\Gamma : \mathbb{G} \to \mathcal{L}_b(\mathcal{H}_0)$ *is said to be of positive-type if for all* $n \in \mathbb{N}$*,* $t_1, \cdots, t_n \in \mathbb{G}$ *and* $x_1, \cdots, x_n \in \mathcal{H}_0$*,*

$$\sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j) x_i, x_j \rangle_{\mathcal{H}_0} \geq 0.$$

It is straightforward to see that a positive-type operator-valued function is hermitian non-negative definite. The other implication is not as easy to prove and will be discussed below. Note that unitary representations are hermitian non-negative definite and therefore Stone's theorem is, indeed, a generalization of Bochner's theorem for a particular type of hermitian non-negative definite operator-valued functions. As a full generalization, the following theorem holds.

Theorem 6.3. *Let* $(\mathbb{G}, +)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a Hilbert space and* $\Gamma : \mathbb{G} \to \mathcal{L}_b(\mathcal{H}_0)$ *continuous for the w.o.t. Then the following propositions are equivalent*

(i) Γ *is hermitian non-negative definite.*

(ii) Γ *is of positive-type.*

(iii) There exists a regular p.o.v.m. ν *on* $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{H}_0)$ *such that*

$$\Gamma(h) = \int \mathrm{e}_h \, \mathrm{d}\nu = \int \chi(h) \, \nu(\mathrm{d}\chi), \quad h \in \mathbb{G}. \tag{6.3}$$

Moreover ν *is the unique regular p.o.v.m. satisfying* (6.3)*.*

These results, as well as Stone's theorem for normal Hilbert modules (see Theorem 2.6) can be proven in different ways, each of which exhibits a specific interest. They also emphasize close relations between these concepts as it turns out that almost every result can be obtained as a consequence of any of the others. As a summary, Figure 1 gives a graphical representation of some interesting implications found in the literature. Arrows with the same color indicate a path of implications usually followed by one or several authors. A few comments on such paths are needed.

- Bochner's and Stone's theorems can be derived on their own using Fourier theory and Riesz-Markov's representation theorem.
- The proofs of Bochner's theorem from Stone's theorem (in cyan) and Naimark's moment theorem from the generalization of Stone's theorem (in brown) use very similar concepts.
- These concepts are closely related to dilation theory (see [61, 3], [23, Section 8]) which is used in [49] to prove Naimark's moment theorem (in green).
- A particular proof of Stone's theorem from Bochner's theorem (in blue) is common in the literature. The proof consists in showing (1.6) when Γ is an u.r. and then proving that the p.o.v.m. ν obtained is actually a projection-valued measure. In fact, the hypothesis that Γ is an u.r. is only useful to show that ν is projection-valued and not to show (1.6). This means that this proof contains a proof of Bochner's theorem for operators as we explicitly represented in blue.
- Concerning the generalization of Bochner's theorem (Theorem 6.3), two results can be found depending on the hypothesis made on the function Γ (hermitian non-negative definite or of positive type). The most general is (i) $\Rightarrow$ (iii) and it is proven (as discussed in the previous point) in a simpler way (without using modules nor dilation theory) than

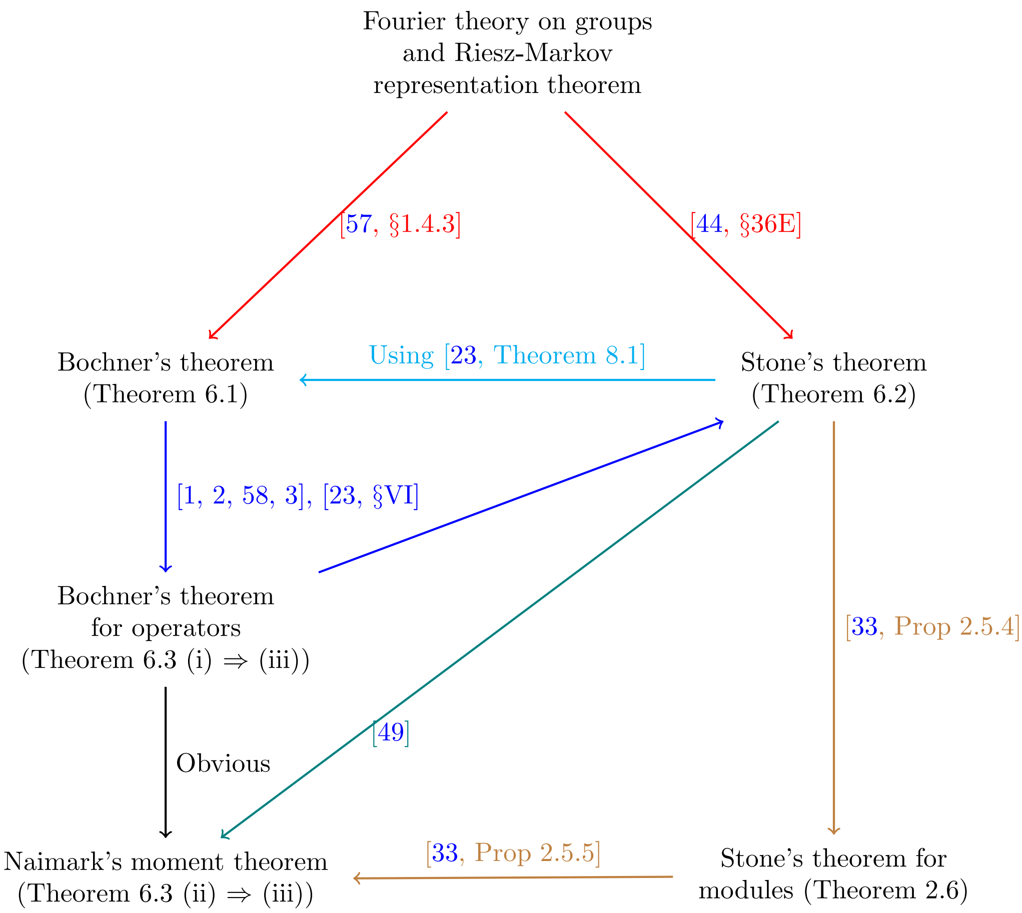

Figure 1: Possible proof paths between the principal results and related concepts.

the other implication ((ii) $\Rightarrow$ (iii)). The converse implications are often omitted or stated without proof and the equivalence of Theorem 6.3 is not common in the literature, but can be found in [3]. The implication (iii) $\Rightarrow$ (i) is easily verified using simple properties of p.o.v.m. but (iii) $\Rightarrow$ (ii) does not seem trivial to show. In [3, Theorem 2], the author provides a proof which makes use of dilation theory. This can be avoided using the fact that, if ν is a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$, then for all $n \in \mathbb{N}^*$, and $x_1, \cdots, x_n \in \mathcal{H}_0$, the mapping

$$\mu : A \mapsto \begin{bmatrix} \langle \nu(A)x_1, x_1 \rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A)x_n, x_1 \rangle_{\mathcal{H}_0} \\ \vdots & \ddots & \vdots \\ \langle \nu(A)x_1, x_n \rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A)x_n, x_n \rangle_{\mathcal{H}_0} \end{bmatrix}$$

defines a p.o.v.m. on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathbb{C}^{n\times n})$ (*i.e.* a hermitian non-negative matrix valued measure). Then, using the results of [54, Section 2] we get that for all $i, j \in [\![1, n]\!]$, $\mu_{i,j} : A \mapsto [\mu(A)]_{i,j}$ admits a density $f_{i,j}$ with respect to the non-negative finite measure $\|\mu\|_1 : A \mapsto \|\mu(A)\|_1 = \mathrm{Tr}(\mu(A))$ and that the matrix-valued function $f : \chi \mapsto (f_{i,j}(\chi))_{1\le i,j\le n}$ is $\|\mu\|_1$-a.e. hermitian, non-negative. Using this, if $\Gamma : h \mapsto \int \chi(h)\, \nu(\mathrm{d}\chi)$, we get for all

$n \in \mathbb{N}^*$, $t_1, \cdots, t_n \in \mathbb{G}$ and $x_1, \cdots, x_n \in \mathcal{H}_0$

$$\begin{aligned}
\sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j)x_i, x_j \rangle_{\mathcal{H}_0} &= \sum_{i,j=1}^{n} \int \chi(t_i)\overline{\chi(t_j)}\, \mu_{i,j}(\mathrm{d}\chi) \\
&= \sum_{i,j=1}^{n} \int \chi(t_i)\overline{\chi(t_j)} f_{i,j}(\chi)\, \|\mu\|_1(\mathrm{d}\chi) \\
&= \int \underbrace{\sum_{i,j=1}^{n} \chi(t_i)\overline{\chi(t_j)} f_{i,j}(\chi)}_{\geq 0\ \|\mu\|_1\text{-a.e.}}\, \|\mu\|_1(\mathrm{d}\chi) \\
&\geq 0\,.
\end{aligned}$$

6.2 An alternative path for constructing spectral representations

In Section 6.1, we saw that Bochner's theorem can be generalized to operator-valued non-negative definite functions. This result can be used to get the same results as in Theorem 3.2 but in a different order. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbb{G}, +)$ an l.c.a. group. Let $\mathcal{H}_0$ be a separable Hilbert space and set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Let $X := (X_t)_{t \in \mathbb{G}}$ be a centered weakly stationary $\mathcal{H}_0$-valued process. Then it is easy to verify that the autocovariance operator function Γ_X is hermitian non-negative definite and continuous for the w.o.t. Hence, by Theorem 6.3, there exists a unique regular p.o.v.m. ν_X of $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{H}_0)$ which satisfies (3.2). Since $\nu_X(\hat{\mathbb{G}}) = \Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$, ν_X is a trace-class p.o.v.m. by Lemma 2.3. Now, call $\mathrm{e}_t : \chi \mapsto \chi(t)$ for all $t \in \mathbb{G}$, then, for all $h, t \in \mathbb{G}$, for all $x, y \in \mathcal{H}_0$,

$$\mathrm{Cov}\left(y^{\mathsf{H}} X_h, x^{\mathsf{H}} X_t\right) = y^{\mathsf{H}}\Gamma(h-t)x = y^{\mathsf{H}}\left(\int \chi(h-t)\nu(\mathrm{d}\chi)\right)x = \left\langle \mathrm{e}_h y^{\mathsf{H}}, \mathrm{e}_t x^{\mathsf{H}} \right\rangle_{\mathsf{L}^2(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{O}(\mathcal{H}_0, \mathbb{C}), \nu_X)}\,.$$

Then, by Proposition 2.5 and Theorem 2.11, there is a unique isometry

$$I : \mathsf{L}^2(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \mathcal{O}(\mathcal{H}_0, \mathbb{C}), \nu_X) \to \mathcal{M}(\Omega, \mathcal{F}, \mathbb{P})$$

which maps $\mathrm{e}_t x^{\mathsf{H}}$ to $x^{\mathsf{H}} X_t$ for all $t \in \mathbb{G}, x \in \mathcal{H}_0$. Using Theorem 2.14, we get that there exists a unique $\mathcal{H}_0$-valued random c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathbb{G}}, \mathcal{B}(\hat{\mathbb{G}}), \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_X such that $I^{\mathbb{C}}_{\hat{X}} = I$ and $\hat{X}$ is regular because ν_X is a regular trace class p.o.v.m.. Finally, for all $t \in \mathbb{G}$ and $x \in \mathcal{H}_0$,

$$x^{\mathsf{H}} X_t = \int \chi(t) x^{\mathsf{H}} \hat{X}(\mathrm{d}\chi) = x^{\mathsf{H}}\left(\int \chi(t)\hat{X}(\mathrm{d}\chi)\right)$$

which implies Relation (3.1).

Remark 6.1. *As mentioned in Remark 1.4, it is interesting to note that, in this proof, we use a milder notion of continuity for Γ_X (continuity for the s.o.t.). In fact, the last part of the proof of Theorem 3.2 shows that, in order to have weak-continuity of Γ_X, it is enough to have Relation* (3.2) *which can be obtained using only continuity for the s.o.t. We can therefore state the two following results*

1. *A hermitian non-negative definite operator-valued function $\Gamma : \mathbb{G} \to \mathcal{L}_b(\mathcal{H}_0)$ such that $\Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$ is weakly continuous if and only if it is continuous for the s.o.t.*
2. *An L^2, $\mathcal{H}_0$-valued process $X = (X_t)_{t \in \mathbb{G}}$ is weakly-stationary if and only if for all $x \in \mathcal{H}_0$, the L^2, complex-valued process $(\langle X_t, x \rangle_{\mathcal{H}_0})_{t \in \mathbb{G}}$ is weakly-stationary.*

6.3 Comparison with recent approaches

Recently, **R1**, **R2** and the problem of defining filtering in the spectral domain have been addressed for the case $\mathbb{G} = \mathbb{Z}$ in [62] under additional assumptions. An attempt at relaxing these assumption was proposed in [16]. We list here and comment the principal results on spectral analysis presented in [62], [16].

About **R1** : With the additional assumption that $\sum_{h\in\mathbb{Z}} \|\Gamma_X(h)\| < +\infty$, [62, Proposition 2.3.5] states that **R1** holds with $\nu_X(\mathrm{d}\lambda) = f_X(\lambda)\mathrm{d}\lambda$ where

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}h\lambda} \Gamma_X(h) \in \mathcal{S}_1^+(\mathcal{H}_0) \; ,$$

where the series converges in $\|\cdot\|$. This result restricts the whole spectral theory to the case where the spectral operator measure admits a density with respect to Lebesgue's measure on $(-\pi, \pi]$ and the existence of such a density is proven under restrictive summability conditions on the autocovariance operator. With this result, we cannot study processes with seasonal components (whose spectral measure have atoms and therefore no density with respect to Lebesgue's measure) or long-memory processes (for which $\sum_{h\in\mathbb{Z}} \|\Gamma_X(h)\| = +\infty$). In [16], **R1** is proved without the summability assumption but the measure ν_X is constructed via compactification of $\mathcal{L}_b^+(\mathcal{H}_0)$. This compactification makes it possible to define "infinite" operator measures which is not necessary here because p.o.v.m.'s theory is sufficient and makes the construction easier as discussed in Section 6.1.

About **R2** : Assuming ν_X has a density f_X with respect to Lebesgue's measure on $(-\pi, \pi]$, such that $f_X \in L^p((-\pi, \pi], \mathcal{B}((-\pi, \pi]), \mathcal{S}_1(\mathcal{H}_0))$ for some $p \in (1, +\infty]$, [62, Theorem 2.4.3] provides the Stieltjes integral representation for all $t \in \mathbb{Z}$,

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Z_\lambda \quad \mathbb{P}\text{-a.e.}, \tag{6.4}$$

where $\lambda \mapsto Z_\lambda$ has orthogonal increments. This result is provided without assuming existence of a density with respect to Lebesgue's measure in [16] and is equivalent to **R2** with $\hat{X}((-\pi, \lambda]) = Z_\lambda$ which becomes a c.a.o.s. measure. In [62, Theorem 2.5.1], the author constructs a space (denoted by $\mathfrak{H}$) similar to $\mathsf{L}^2((-\pi, \pi], \mathcal{B}((-\pi, \pi]), \mathcal{O}(\mathcal{H}_0), \nu_X)$ and proves the isometric property of the spectral representation. The difference with the results we present in Sections 2 and Section 3 is that, by making the module structure of $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ explicit, we believe that the construction is clearer and more complete. In particular, the spectral representation $\hat{X}$ is a c.a.g.o.s. measure, which extends the c.a.o.s. property, and the time and spectral domains are shown to be gramian-isometrically isomorphic. In addition, the space $\mathfrak{H}$ of [62] is constructed as the completion of a pre-Hilbert space, that is a quotient space of Cauchy sequences, which provides little intuition on the space of transfer operator functions one can consider for filtering X. On the contrary, we provide a definition of the space of operator-valued functions $\mathsf{L}^2((-\pi, \pi], \mathcal{B}((-\pi, \pi]), \mathcal{O}(\mathcal{H}_0), \nu_X)$ which is simple to characterize.

A Useful functional analysis results

A.1 Diagonalization of compact positive operators

Let $\mathcal{H}_0$ be a separable Hilbert space and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$. Then $s \in \mathbb{C}$ is said to be an *eigenvalue* of P if $\ker(\mathrm{P} - s\mathrm{Id}) \neq \{0\}$. If s is an eigenvalue of P, we say that $\ker(\mathrm{P} - s\mathrm{Id})$ is the associated *eigensubspace* and its dimension is called the *multiplicity* of s. We denote by $\mathrm{spec_p}(\mathrm{P})$ the set of eigenvalues of P (called the *point spectrum* of P). P is said to be *diagonalizable* if $\mathcal{H}_0$ has a Hilbert-basis of eigenvectors of P. If $\mathrm{P} \in \mathcal{K}(\mathcal{H}_0)$ and is auto-adjoint, then it is diagonalizable and $\mathrm{spec_p}(\mathrm{P})$ is at most discrete, every non-zero eigenvalue has finite dimension and eigensubspace associated to different eigenvalues are orthogonal. We denote by $N_{\mathrm{sp}}(\mathrm{P})$ the cardinal of $\mathrm{spec_p}(\mathrm{P})$ which is finite if and only if $\mathrm{rank}(\mathrm{P}) < +\infty$ and if not, then $\mathrm{spec_p}(\mathrm{P})$ admits 0 as its unique accumulation point (equivalently, this means that any way of representing the elements of $\mathrm{spec_p}(\mathrm{P})$ gives a sequence converging to 0).

In order to have a representation which takes into account both cases we add zeros at the end of the sequence in the case where $N_{\mathrm{sp}}(\mathrm{P}) < +\infty$. This allows us to systematically represent the eigenvalues of P as a sequence converging to 0. When $\mathrm{P} \in \mathcal{K}^+(\mathcal{H}_0)$ all its eigenvalues are non-negative and it is convenient to represent them in decreasing order which, in the case where $N_{\mathrm{sp}}(\mathrm{P}) = +\infty$, gives a sequence of strictly positive numbers decreasing to 0 even if $0 \in \mathrm{spec_p}(\mathrm{P})$. We will denote by $(\mathrm{s}_i(\mathrm{P}))_{i\in\mathbb{N}}$ such a sequence of distinct eigenvalues, that is if $N_{\mathrm{sp}}(\mathrm{P}) < +\infty$, then $\mathrm{s}_0(\mathrm{P}) > \mathrm{s}_1(\mathrm{P}) > \cdots > \mathrm{s}_{N_{\mathrm{sp}}(\mathrm{P})}(\mathrm{P}) > 0$ and $\mathrm{s}_i(\mathrm{P}) = 0$ for all $i > N_{\mathrm{sp}}(\mathrm{P})$ and if $N_{\mathrm{sp}}(\mathrm{P}) = +\infty$, then $\mathrm{s}_0(\mathrm{P}) > \mathrm{s}_1(\mathrm{P}) > \cdots > 0$. In the latter case, one need to keep in mind the fact that 0 can be an eigenvalue even if it is not represented in the sequence. Using this representation $(\mathrm{s}_i(\mathrm{P}))_{i\in\mathbb{N}}$, we will also denote by $m_i(\mathrm{P})$ the multiplicity of $\mathrm{s}_i(\mathrm{P})$ and by $\Pi_i(\mathrm{P})$ the orthogonal projection onto $\ker(\mathrm{P}-\mathrm{s}_i(\mathrm{P})\mathrm{Id})$ for all $i\in\mathbb{N}$. Finally we define $(\alpha_i(\mathrm{P}))_{i\in\mathbb{N}}$ the piecewise constant sequence obtained by repeating the values of $(\mathrm{s}_i(\mathrm{P}))_{i\in\mathbb{N}}$ as often as their multiplicities. With these notations we can write

$$\mathrm{P}=\sum_{i\in\mathbb{N}}\mathrm{s}_i(\mathrm{P})\Pi_i(\mathrm{P}) \tag{A.1}$$

where the series converges in operator norm, and if $\mathrm{P}\neq 0$,

$$\mathrm{Id}=P_{\ker(\mathrm{P})}+P_{\overline{\mathrm{Im}(\mathrm{P})}}=P_{\ker(\mathrm{P})}+\sum_{0\le i<N_{\mathrm{sp}}(\mathrm{P})}\Pi_i \tag{A.2}$$

where, if $N_{\mathrm{sp}}(\mathrm{P})=+\infty$, the series converges in s.o.t. (If $\mathrm{P}=0$ we have $\mathrm{Id}=P_{\ker(\mathrm{P})}=\Pi_i$ for all $i\in\mathbb{N}$). Moreover the following measurability properties hold (recall that the notion of simple measurability is defined in Section 2.1 and $\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0))=\big\{A\cap\mathcal{K}^+(\mathcal{H}_0)\,:\,A\in\mathcal{B}(\mathcal{K}(\mathcal{H}_0))\big\}$).

Proposition A.1. *The following assertions hold for all $i\in\mathbb{N}$.*

(i) $\alpha_i:\mathrm{P}\mapsto\alpha_i(\mathrm{P})$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$.

(ii) $m_i:\mathrm{P}\mapsto m_i(\mathrm{P})$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\overline{\mathbb{N}},\mathcal{P}(\overline{\mathbb{N}}))$.

(iii) $\mathrm{s}_i:\mathrm{P}\mapsto\mathrm{s}_i(\mathrm{P})$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$.

(iv) $\mathrm{rank}:\mathrm{P}\mapsto\mathrm{rank}(\mathrm{P})$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\overline{\mathbb{N}},\mathcal{P}(\overline{\mathbb{N}}))$.

(v) $N_{\mathrm{sp}}:\mathrm{P}\mapsto N_{\mathrm{sp}}(\mathrm{P})$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\overline{\mathbb{N}},\mathcal{P}(\overline{\mathbb{N}}))$.

(vi) $\Pi_i:\mathrm{P}\mapsto\Pi_i(\mathrm{P})$ *is simply measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $\mathcal{L}_b(\mathcal{H}_0)$.

(vii) $\mathrm{P}\mapsto P_{\ker(\mathrm{P})}$ *is simply measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $\mathcal{L}_b(\mathcal{H}_0)$.

(viii) There exists a family $(\psi_i)_{i\in\mathbb{N}}$ *of functions* $\psi_i:\mathrm{P}\mapsto\psi_i(\mathrm{P})$ *which are measurable from* $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\mathcal{H}_0,\mathcal{B}(\mathcal{H}_0))$ *such that for all* $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, $(\psi_i(\mathrm{P}))_{i\in\mathbb{N}}$ *is orthonormal and for all* $i\in\mathbb{N}$, $\psi_i(\mathrm{P})\in\ker(\mathrm{P}-\mathrm{s}_i(\mathrm{P})\mathrm{Id})$.

Proof. We follow the ideas of the proofs of [45, Theorem 2.10] and [33, Lemma 3.4.7].
Proof of (i). By [33, Lemma 3.4.6], for all $n\in\mathbb{N}$ and all $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$,

$$\sum_{i=0}^{n}\alpha_i(\mathrm{P})=\max\left\{\sum_{i=0}^{n}\langle\mathrm{P}x_i,x_i\rangle_{\mathcal{H}_0}\;:\;(x_0,\cdots,x_n)\text{ is orthonormal}\right\}$$

and therefore $\sum_{i=0}^{n}\alpha_i$ is measurable from $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ to $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Then using $\alpha_i=\sum_{j=1}^{i}\alpha_j-\sum_{j=1}^{i-1}\alpha_j$ we get measurability of α_i for all $i\in\mathbb{N}$.
Proof of (ii). By definition, for all $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, $m_0(\mathrm{P})=\inf\{j\in\mathbb{N}\,:\,\alpha_j(\mathrm{P})\neq\alpha_{j+1}(\mathrm{P})\}$ with the convention $\inf\emptyset=+\infty$ and for all $i\in\mathbb{N}^*$,

$$m_i(\mathrm{P})=\begin{cases}\inf\{j>m_{i-1}(\mathrm{P})\,:\,\alpha_j(\mathrm{P})\neq\alpha_{j+1}(\mathrm{P})\}-m_{i-1}(\mathrm{P}) & \text{if } m_{i-1}(\mathrm{P})<+\infty\\ +\infty & \text{otherwise}\end{cases}.$$

Measurability of the m_i's is then proven by induction.
Proof of (iii). For all $i\in\mathbb{N}$, for all $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, $\mathrm{s}_i(\mathrm{P})=\alpha_{m_i(\mathrm{P})}(\mathrm{P})\mathbb{1}_{\{m_i(\mathrm{P})\neq 0\}}$ hence s_i is measurable.
Proof of (iv). For all $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, $\mathrm{rank}(\mathrm{P})=\sum_{i\in\mathbb{N}}\mathbb{1}_{\{\alpha_i(\mathrm{P})\neq 0\}}$ hence rank is measurable.
Proof of (v). For all $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, $N_{\mathrm{sp}}(\mathrm{P})=\sum_{i\in\mathbb{N}}\mathbb{1}_{\{\mathrm{s}_i(\mathrm{P})\neq 0\}}$ hence N_{sp} is measurable.
Proof of (vi). Let $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, then from (A.1) one can show that for all $n\in\mathbb{N}$,

$$\left(\frac{\mathrm{P}}{\mathrm{s}_0(\mathrm{P})}\right)^n=\sum_{k\in\mathbb{N}}\left(\frac{\mathrm{s}_k(\mathrm{P})}{\mathrm{s}_0(\mathrm{P})}\right)^n\Pi_k(\mathrm{P})\quad\text{in s.o.t.}$$

and for all $1\le i<N_{\mathrm{sp}}(\mathrm{P})$,

$$\left(\frac{\mathrm{P}-\sum_{k=0}^{i-1}\mathrm{s}_k\Pi_k}{\mathrm{s}_i}\right)^n=\sum_{k\in\mathbb{N}}\left(\frac{\mathrm{s}_k(\mathrm{P})}{\mathrm{s}_i(\mathrm{P})}\right)^n\Pi_k\quad\text{in s.o.t.}$$

From these two relations and (A.2) we easily get

$$\Pi_0(\mathrm{P}) = \lim_{n\to+\infty}\left(\frac{\mathrm{P}}{\mathrm{s}_0(\mathrm{P})}\right)^n \mathbb{1}_{\{\mathrm{s}_0(\mathrm{P})\neq 0\}} + \mathrm{Id}\mathbb{1}_{\{\mathrm{s}_0(\mathrm{P})=0\}}$$

and for all $i \geq 1$,

$$\begin{aligned}\Pi_i(\mathrm{P}) &= \mathrm{Id}\mathbb{1}_{\{\mathrm{s}_0(\mathrm{P})=0\}}\\ &+ \mathbb{1}_{\{\mathrm{s}_0(\mathrm{P})\neq 0\}}\mathbb{1}_{\left\{i<N_{\mathrm{sp}}(\mathrm{P})\right\}}\lim_{n\to+\infty}\left(\frac{\mathrm{P}-\sum_{k=0}^{i-1}\mathrm{s}_k(\mathrm{P})\Pi_k(\mathrm{P})}{\mathrm{s}_i(\mathrm{P})}\right)^n\\ &+ \mathbb{1}_{\{\mathrm{s}_0(\mathrm{P})\neq 0\}}\mathbb{1}_{\left\{i\geq N_{\mathrm{sp}}(\mathrm{P})\right\}}\left(\mathrm{Id}-\sum_{0\leq k<N_{\mathrm{sp}}(\mathrm{P})}\Pi_k(\mathrm{P})\right)\end{aligned}$$

where the convergences are in s.o.t. Hence by measurability of the s_i's and of N_{sp} we get by induction than the Π_i's are simply measurable as limit in s.o.t. of simply measurable functions.

Proof of (vii). Simple measurability of $\mathrm{P}\mapsto P_{\ker(\mathrm{P})}$ comes from (A.2), simple measurability of the Π_i's and measurability of N_{sp}.

Proof of (viii). Let $(\phi_n)_{n\in\mathbb{N}}$ be a Hilbert-basis of $\mathcal{H}_0$, then define for all

$$\tau_i : \mathrm{P} \mapsto \begin{cases}\min\left\{n\in\mathbb{N} \,:\, \Pi_i(\mathrm{P})\phi_n \neq 0\right\} & 0\leq i<N_{\mathrm{sp}}(\mathrm{P})\\ \min\left\{n>\tau_{i-1}(\mathrm{P}) \,:\, P_{\ker(\mathrm{P})}\phi_n\neq 0\right\} & i\geq N_{\mathrm{sp}}(\mathrm{P})\end{cases}.$$

Note that τ_i never takes the value $+\infty$ because for all $i\in\mathbb{N}$, $\Pi_i(\mathrm{P})\neq 0$ and if $N_{\mathrm{sp}}(\mathrm{P})<+\infty$, then $\ker(\mathrm{P})$ has infinite dimension and therefore there are infinitely many $n\in\mathbb{N}$ such that $P_{\ker(\mathrm{P})}\phi_n\neq 0$. Then measurability of N_{sp} and simple measurability of the Π_i's give that the τ_i's are measurable from $(\mathcal{K}^+(\mathcal{H}_0),\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ to $(\mathbb{N},\mathcal{P}(\mathbb{N}))$. Now define for all $i\in\mathbb{N}$, $\varphi_i : \mathrm{P}\mapsto \Pi_i(\mathrm{P})\phi_{\tau_i(\mathrm{P})}$ and the sequence $(\tilde{\psi}_i)_{i\in\mathbb{N}}$ obtained by applying the Gram-Schmidt algorithm to the φ_i's, that is $\tilde{\psi}_0 : \mathrm{P}\mapsto\varphi_0(\mathrm{P})$ and for all $i\geq 1$,

$$\tilde{\psi}_i : \mathrm{P}\mapsto \varphi_i(\mathrm{P}) - \sum_{k=0}^{i-1}\frac{\left\langle\varphi_i(\mathrm{P}),\tilde{\psi}_k(\mathrm{P})\right\rangle}{\left\|\tilde{\psi}_k(\mathrm{P})\right\|_{\mathcal{H}_0}^2}\tilde{\psi}_k(\mathrm{P}) .$$

Finally, define for all $i\in\mathbb{N}$, $\psi_i : \mathrm{P}\mapsto \tilde{\psi}_i(\mathrm{P})/\left\|\tilde{\psi}_i(\mathrm{P})\right\|_{\mathcal{H}_0}$. Then, measurability of the φ_i's implies measurability of the ψ_i's and, by construction for all $\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$, the family $(\psi_i(\mathrm{P}))_{i\in\mathbb{N}}$ is orthonormal.

□

A.2 Singular values decomposition

Let $\mathcal{H}_0$, $\mathcal{G}_0$ be two separable Hilbert spaces and $\mathrm{P}\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$, then $\mathrm{P}^{\mathsf{H}}\mathrm{P}\in\mathcal{K}^+(\mathcal{H}_0)$ and $\mathrm{P}\mathrm{P}^{\mathsf{H}}\in\mathcal{K}^+(\mathcal{G}_0)$ and these two operators have the same non-zero eigenvalues with the same (finite) multiplicities. Define the set $\mathrm{sing}(\mathrm{P})$ of *singular values* of $\mathrm{P}\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$ as

$$\mathrm{sing}(\mathrm{P}) = \left\{\mathrm{s}^{1/2} \,:\, \mathrm{s}\in\mathrm{spec}_{\mathrm{p}}(\mathrm{P}^{\mathsf{H}}\mathrm{P})\setminus\{0\}\right\} = \left\{\mathrm{s}^{1/2} \,:\, \mathrm{s}\in\mathrm{spec}_{\mathrm{p}}(\mathrm{P}\mathrm{P}^{\mathsf{H}})\setminus\{0\}\right\}$$

and for all $\sigma\in\mathrm{sing}(\mathrm{P})$ we call *multiplicity* of σ the multiplicity of σ^2 as an eigenvalue of $\mathrm{P}^{\mathsf{H}}\mathrm{P}$ or $\mathrm{P}^{\mathsf{H}}\mathrm{P}$ (which are the same). The well-known singular value decomposition theorem can then be stated as follows.

Theorem A.2 (Singular value decomposition)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and $\mathrm{P}\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$ then there exist two Hilbert basis $(\phi_n)_{0\leq n<\mathrm{rank}(\mathrm{P})}$ and $(\psi_n)_{0\leq n<\mathrm{rank}(\mathrm{P})}$ of $\overline{\mathrm{Im}(\mathrm{P}^{\mathsf{H}})}$ and $\overline{\mathrm{Im}(\mathrm{P})}$ respectively and $(\sigma_n)_{0\leq n<\mathrm{rank}(\mathrm{P})}$ representing the elements of $\mathrm{sing}(\mathrm{P})$ repeated as often as their multiplicity such that*

$$\mathrm{P} = \sum_{0\leq n<\mathrm{rank}(\mathrm{P})}\sigma_n\psi_n\otimes\phi_n \tag{A.3}$$

where the series converges in operator norm. Moreover, if $\mathrm{rank}(\mathrm{P})=+\infty$, $\lim_{n\to+\infty}\sigma_n=0$.

Similarly to the eigendecomposition, the singular values are usually written as a decreasing sequence $(\sigma_i(\mathrm{P}))_{0\leq i<\mathrm{rank}(\mathrm{P})}$.

A.3 Generalized inverse of an operator

Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, then the mapping

$$\mathrm{P}_{|\ker(\mathrm{P})^\perp \to \mathrm{Im}(\mathrm{P})} : \begin{array}{ccc} \ker(\mathrm{P})^\perp & \to & \mathrm{Im}(\mathrm{P}) \\ x & \mapsto & \mathrm{P}x \end{array}$$

is an isomorphism and we define $\mathrm{P}^\dagger \in \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0)$ (called the *generalized inverse* of P) as the linear extension of $\left(\mathrm{P}_{|\ker(\mathrm{P})^\perp \to \mathrm{Im}(\mathrm{P})}\right)^{-1}$ to $\mathcal{D}(\mathrm{P}^\dagger) := \mathrm{Im}(\mathrm{P}) \oplus \mathrm{Im}(\mathrm{P})^\perp$ such that $\ker(\mathrm{P}^\dagger) = \mathrm{Im}(\mathrm{P})^\perp$. In other words, for all $x \in \mathcal{D}(\mathrm{P}^\dagger)$, there exists $(x_1, x_2) \in \mathrm{Im}(\mathrm{P}) \times \mathrm{Im}(\mathrm{P})^\perp$ such that $x = x_1 + x_2$, then $\mathrm{P}^\dagger x = \left(\mathrm{P}_{|\ker(\mathrm{P})^\perp \to \mathrm{Im}(\mathrm{P})}\right)^{-1} x_1$.

The subspace $\mathcal{D}(\mathrm{P})$ is dense in $\mathcal{G}_0$ and is equal to $\mathcal{G}_0$ if and only if $\mathrm{Im}(\mathrm{P})$ is closed, in which case $\mathrm{P}^\dagger \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{H}_0)$. The operators P and $\mathrm{P}^\dagger$ are linked by the relation

$$\mathrm{P}^\dagger \mathrm{P} = P_{\ker(\mathrm{P})^\perp} \tag{A.4}$$

and it is easy to show that, if $\mathrm{Q} \in \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0)$, then $\mathrm{Q} = \mathrm{P}^\dagger$ if and only if $\mathrm{QP} = P_{\ker(\mathrm{P})^\perp}$ and $\mathrm{Q}_{|\ker(\mathrm{P})^\perp} = 0$.

The identity (A.4), along with the fact that a projection is compact if and only if it has finite rank, gives (see [25, Theorem 3.1.3]) that a compact operator has closed range if and only if it has finite rank. The operator $\mathrm{PP}^\dagger$ is not as easy to characterize but when $\mathrm{Im}(\mathrm{P})$ is closed, we have $\mathrm{PP}^\dagger = P_{\mathrm{Im}(\mathrm{P})}$. Finally, in the case where $\mathrm{P} \in \mathcal{K}^+(\mathcal{H}_0)$, the generalized inverse can be diagonalized as follows.

Proposition A.3. *Let $\mathcal{H}_0$ be a separable Hilbert space and $\mathrm{P} \in \mathcal{K}^+(\mathcal{H}_0)$, then, defining for all $i \in \mathbb{N}$, $\mathrm{s}_i^\dagger(\mathrm{P}) = 1/\mathrm{s}_i(\mathrm{P})$ if $\mathrm{s}_i(\mathrm{P}) \neq 0$ and 0 otherwise, we get*

$$\mathcal{D}(\mathrm{P}^\dagger) = \left\{ x \in \mathcal{H}_0 \;:\; \sum_{i\in\mathbb{N}} \left(\mathrm{s}_i^\dagger(\mathrm{P})\right)^2 \|\Pi_i(\mathrm{P})x\|^2_{\mathcal{H}_0} < +\infty \right\} \tag{A.5}$$

and for all $x \in \mathcal{D}(\mathrm{P}^\dagger)$,

$$\mathrm{P}^\dagger x = \sum_{i\in\mathbb{N}} \mathrm{s}_i^\dagger(\mathrm{P})\Pi_i(\mathrm{P})x \tag{A.6}$$

Proof. The inclusion ($\subset$) in (A.5) is straightforward. To show the converse inclusion, let $x \in \mathcal{H}_0$ such that $\sum_{i\in\mathbb{N}} \left(\mathrm{s}_i^\dagger(\mathrm{P})\right)^2 \|\Pi_i(\mathrm{P})x\|^2_{\mathcal{H}_0} < +\infty$, then $y := \sum_{i\in\mathbb{N}} \mathrm{s}_i(\mathrm{P})^\dagger \Pi_i(\mathrm{P})x$ exists because the series converges in $\mathcal{H}_0$. Now, we write $x = P_{\overline{\mathrm{Im}(\mathrm{P})}}x + P_{\mathrm{Im}(\mathrm{P})^\perp x}$ with

$$P_{\overline{\mathrm{Im}(\mathrm{P})}}x = \sum_{0\le i<N_{\mathrm{sp}}(\mathrm{P})} \Pi_i(\mathrm{P})x = \sum_{0\le i<N_{\mathrm{sp}}(\mathrm{P})} \mathrm{s}_i(\mathrm{P})\mathrm{s}_i^\dagger(\mathrm{P})\Pi_i(\mathrm{P})x = \mathrm{P}y \in \mathrm{Im}(\mathrm{P}) ,$$

and therefore $x \in \mathcal{D}(\mathrm{P}^\dagger)$ which concludes the proof of (A.5).

To show (A.6), let $x \in \mathcal{D}(\mathrm{P}^\dagger)$ and define the operator

$$\mathrm{Q} : \begin{array}{ccc} \mathcal{D}(\mathrm{P}^\dagger) & \to & \mathcal{H}_0 \\ x & \mapsto & \sum_{i\in\mathbb{N}} \mathrm{s}_i^\dagger(\mathrm{P})\Pi_i(\mathrm{P})x \end{array}$$

Then, it is easy to verify that $\mathrm{QP} = P_{\overline{\mathrm{Im}(\mathrm{P})}} = P_{\ker(\mathrm{P})^\perp}$ and that $\mathrm{Q}_{|\mathrm{Im}(\mathrm{P})^\perp} = 0$ which imply that $\mathrm{Q} = \mathrm{P}^\dagger$. $\square$

Using this result, we can show the useful measurability property.

Corollary A.4. *Let $\mathcal{H}_0$ be a separable Hilbert space. Then the mapping $\mathrm{P} \mapsto \mathrm{P}^\dagger$ is $\mathcal{O}$-measurable (see Section 2.1 for the definition) from $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0))$ to $\mathcal{O}(\mathcal{H}_0)$*

Proof. Measurability of the s_i's and simple measurability of the Π_i's shown in Proposition A.1 give Condition (i) of the definition of $\mathcal{O}$-measurability using (A.5) and Condition (ii) using (A.6). $\square$

B Additional results on vector and operator valued measures

B.1 Projection-valued and gramian-projection-valued measures

Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ a separable Hilbert space. A *projection-valued measure* (p.v.m.) ξ on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is a p.o.v.m. valued in the space of orthogonal projections on $\mathcal{H}_0$. If in addition $\xi(\Lambda) = \mathrm{Id}$, we say that ξ is *normalized*. This notion appears in diagonalization of non-compact operators and in Stone's theorem where such measures are often mentioned as "spectral measures" or "spectral operator measures" (see e.g. [12, Chapter IX]) but it must not be mistaken with what we defined as "spectral operator measures" for weakly stationary stochastic process. When working with modules, the notion of p.v.m.'s can be extended to *gramian-projection-valued measures* (g.p.v.m.) which play the same role as p.v.m.'s for the extension of Stone's theorem on modules. If $\mathcal{H}$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, then a p.v.m. ξ on $(\Lambda, \mathcal{A}, \mathcal{H})$ is said to be a g.p.v.m. if for all $A \in \mathcal{A}$, $\xi(A)$ is a gramian-projection. The notion of regularity used for p.v.m.'s and g.p.v.m.'s is the one defined in Section 2.2 for p.o.v.m.'s.

B.2 Countably additive orthogonally scattered measures

Let $\mathcal{H}_0$ be a Hilbert space and $(\Lambda, \mathcal{A})$ a measurable space. A countably additive orthogonally scattered (c.a.o.s.) measure W on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is an $\mathcal{H}_0$-valued measure which satisfies for all $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$, $\langle W(A), W(B) \rangle_{\mathcal{H}_0} = 0$. The proofs of the following assertions are straightforward.

(i) If W is a c.a.o.s. measure on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$, then $\eta_W : A \mapsto \|W(A)\|^2_{\mathcal{H}_0}$ is a finite, non-negative measure on $(\Lambda, \mathcal{A})$ called the *intensity measure* of W. It satisfies for all $A, B \in \mathcal{A}$,
$$\eta_W(A \cap B) = \langle W(A), W(B) \rangle_{\mathcal{H}_0} \; .$$

(ii) Conversely, if $W : \mathcal{A} \to \mathcal{H}_0$ is such that there exists a finite, non-negative measure η on $(\Lambda, \mathcal{A})$ satisfying $\forall A, B \in \mathcal{A}$, $\langle W(A), W(B) \rangle_{\mathcal{H}_0} = \eta(A \cap B)$. Then W is a c.a.o.s. measure on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with intensity measure η.

When Λ is a locally-compact topological space then, by definition of the intensity measure, we get that a c.a.o.s. measure W is regular (in the sense recalled in Section 2) if and only if its intensity measure is regular. Since we do not assume that a c.a.o.s. measure has finite variation, we cannot use Bochner's integration theory recalled in Section 2.1. However, Assertion (i) implies that we can linearly, continuously and isometrically extend the mapping $\mathbb{1}_A \mapsto W(A)$ to
$$\overline{\mathrm{Span}} \left(1_A \, , \; A \in \mathcal{A} \right) = L^2 \left(\Lambda, \mathcal{A}, \eta_W \right) \; .$$
That is, there exists a unique isometric operator $I_W : L^2(\Lambda, \mathcal{A}, \eta_W) \to \mathcal{H}_0$ such that $\forall A \in \mathcal{A}$, $I_W(\mathbb{1}_A) = W(A)$. Moreover, I_W is unitary from $L^2(\Lambda, \mathcal{A}, \eta_W)$ to $\overline{\mathrm{Span}}^{\mathcal{H}_0} \left(W(A), A \in \mathcal{A} \right)$ and we define integration of $L^2(\Lambda, \mathcal{A}, \eta_W)$ functions with respect to W by setting, for all $f \in L^2(\Lambda, \mathcal{A}, \eta_W)$,
$$\int f \mathrm{d}W := I_W(f) \; .$$
Conversely, if η is a finite, non-negative measure on $(\Lambda, \mathcal{A})$ and I is an isometry from $L^2(\Lambda, \mathcal{A}, \eta)$ to $\mathcal{H}_0$, then there exists a unique c.a.o.s. measure W on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with intensity measure η such that, for all $f \in L^2(\Lambda, \mathcal{A}, \eta)$,
$$w(f) = \int f \, \mathrm{d}W \; .$$

B.3 On the completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$

In Section 2.5, we have defined the normal pre-Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ of square-integrable bounded-operator-valued functions. In the univariate case, this corresponds to $L^2(\Lambda, \mathcal{A}, \nu_X)$ which is a Hilbert space. In the multivariate case, where $\mathcal{H}_0$ and $\mathcal{G}_0$

have finite dimensions, the completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is proven in [54]. However completeness is not guaranteed in the infinite dimensional case, see [45], where the authors refer to [41] for a counter-example. We complete this line of thoughts by providing a necessary and sufficient condition for the completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ in the general case.

Theorem B.1. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is complete for the gramian defined in Proposition 2.8 if and only if* $\mathrm{rank}(f)$ *is finite $\|\nu\|_1$-a.e. In this case,* $\mathrm{rank}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)$ *is finite μ-a.e. for all finite non-negative measure μ which dominates $\|\nu\|_1$.*

Proof. The proof of the fact that we can take μ instead of $\|\nu\|_1$ uses the same arguments we used to prove Relation (2.12) and will be omitted. Now, let us consider that f is a representing function of the density which is in $\mathcal{S}_1^+(\mathcal{H}_0)$ everywhere and let $A := \{\mathrm{rank} f < +\infty\}$ which is in $\mathcal{A}$ by measurability of the rank (see Proposition A.1) and of f. Then by [25, Theorem 3.1.3], we have $A = \{\mathrm{Im} f^{1/2} \text{ is closed}\}$. We show successively that $\|\nu\|_1(A^c) = 0$ is a necessary condition for completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and then that it is sufficient.

Proof of necessity. Suppose that $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is complete and that $\|\nu\|_1(A^c) \neq 0$. Then in order to get a contradiction, we will follow the following two steps.

Step 1 Construct a function $\Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that

$$\text{for all } \lambda \in A^c, \quad \Psi(\lambda) \notin \left\{\mathrm{P} f(\lambda)^{1/2} \;:\; \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)\right\} . \tag{B.1}$$

Step 2 Construct a sequence $(\Phi_n)_{n\in\mathbb{N}} \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)^{\mathbb{N}}$ such that $\Phi_n f^{1/2}$ converges to Ψ in $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$.

Let us explain why these two steps lead to a contradiction. **Step 2** implies that $(\Phi_n f^{1/2})_{n\in\mathbb{N}}$ is Cauchy in $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ which, by the gramian-isometric property shown in Proposition 2.8, means that $(\Phi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Since we assumed completeness, there exists $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that Φ_n converges to Φ in this space, which, again by Proposition 2.8, means that $\Phi_n f^{1/2}$ converges to $\Phi f^{1/2}$ in $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ and thus $\Phi f^{1/2} = \Psi$ $\|\nu\|_1$-a.e. contradicting (B.1).

We now provide the constructions previously described.

Step 1 By Proposition A.1 and composition of measurable functions, we know that the functions $\lambda \mapsto \mathrm{s}_i(\lambda)$ are measurable where $\mathrm{s}_i(\lambda)$ is the i-th eigenvalue of $f(\lambda)^{1/2}$ (in decreasing order with the convention of Appendix A.1). Moreover, Proposition A.1 (and again composition of measurable functions) also gives that there exists a family of measurable functions $(\psi_i)_{i\in\mathbb{N}}$ from Λ to $\mathcal{H}_0$ such that for all $\lambda \in \Lambda$, $(\psi_i(\lambda))_{i\in\mathbb{N}}$ is an orthonormal sequence of eigenvectors associated to the eigenvalues $(\mathrm{s}_i)_{i\in\mathbb{N}}$. Define

$$y : \lambda \mapsto \sum_{n\in\mathbb{N}} \ell_n(\lambda)\psi_n(\lambda)$$

with $\ell_n(\lambda) = C(\lambda)^{-1}\mathrm{s}_n(\lambda)$ where $C(\lambda) = \left(\sum_{n\in\mathbb{N}} \mathrm{s}_n(\lambda)^2\right)^{1/2}$ so that $\|y(\lambda)\|_{\mathcal{H}_0} = 1$. And let

$$\Psi : \lambda \mapsto u \otimes y(\lambda)$$

where $u \in \mathcal{G}_0$ with $\|u\|_{\mathcal{G}_0} = 1$. Then $\Psi \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ because for all $\lambda \in \Lambda$, $\|\Psi(\lambda)\|_2 = 1$.

We conclude by reasoning by contradiction. Suppose that (B.1) does not hold and take $\lambda \in A^c$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ such that $\Psi(\lambda) = \mathrm{P} f(\lambda)^{1/2}$. Then we have that $y(\lambda) \otimes u = \Psi(\lambda)^{\mathsf{H}} = f(\lambda)^{1/2}\mathrm{P}^{\mathsf{H}}$ and thus

$$y(\lambda) = (y(\lambda) \otimes u)(u) = f(\lambda)^{1/2}\mathrm{P}^{\mathsf{H}} u \in \mathrm{Im}(f(\lambda)^{1/2}) .$$

This means that there exists $x \in \mathcal{H}_0$ such that $y(\lambda) = f(\lambda)^{1/2}x$ and we get for all $n \in \mathbb{N}$,

$$C(\lambda)^{-1}\mathrm{s}_n(\lambda) = \ell_n(\lambda) = \left\langle f(\lambda)^{1/2}x, \psi_n(\lambda)\right\rangle_{\mathcal{H}_0} = \left\langle x, f(\lambda)^{1/2}\psi_n(\lambda)\right\rangle_{\mathcal{H}_0} = \mathrm{s}_n(\lambda)\left\langle x, \psi_n(\lambda)\right\rangle_{\mathcal{H}_0} .$$

In particular $\mathrm{s}_n(\lambda) > 0$ implies $\langle x, \psi_n(\lambda)\rangle_{\mathcal{H}_0} = C(\lambda)^{-1}$. Since $\mathrm{rank} f(\lambda) = +\infty$, we know that $\mathrm{s}_n(\lambda) > 0$ for all $n \in \mathbb{N}$ and thus get that $\|x\|_{\mathcal{H}_0} = +\infty$, which is impossible.

Step 2 Define

$$\Phi_n : \lambda \mapsto C(\lambda)^{-1} u \otimes \sum_{k=0}^{n} \psi_k(\lambda) \; .$$

Then $\Phi_n \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ and $\Phi_n(\lambda) f^{1/2}(\lambda) = u \otimes \sum_{k=0}^{n} \ell_k(\lambda)\psi_k(\lambda)$. Then for all $\lambda \in \Lambda$,

$$\left\| \Psi(\lambda) - \Phi_n(\lambda) f^{1/2}(t) \right\|_2^2 = \sum_{k=n+1}^{+\infty} \ell_k(\lambda)^2 \; ,$$

which tends to 0 as $n \to +\infty$ and is bounded by 1. Hence by Lebesgue's dominated converge theorem

$$\int \left\| \Psi - \Phi_n f^{1/2} \right\|_2^2 \mathrm{d}\|\nu\|_1 \xrightarrow[n \to +\infty]{} 0 \; .$$

Proof of sufficiency. Suppose that $\|\nu\|_1(A^c) = 0$, *i.e.* that $\mathrm{Im} f^{1/2}$ is closed $\|\nu\|_1$-a.e. and consider that $f^{1/2}$ is a representing function of the density which has closed range everywhere. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and define for all $n \in \mathbb{N}$, $\Psi_n = \Phi_n f^{1/2}$. Then, by Proposition 2.8, $(\Psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ which is complete, hence $\Psi = \lim_{n\to+\infty} \Psi_n$ exists in $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$. This implies that there exists a subsequence $(\Psi_{\phi(n)})_{n\in\mathbb{N}}$ of $(\Psi_n)_{n\in\mathbb{N}}$ which converges $\|\nu\|_1$-a.e. to Ψ. More explicitly, there exists $B \in \mathcal{A}$, with $\|\nu\|_1(B^c) = 0$ such that $\Psi_{\phi(n)}(\lambda) \xrightarrow[n\to+\infty]{} \Psi(\lambda)$ for all $\lambda \in B$. Let $\lambda \in B$, then for all $x \in \mathcal{G}_0$,

$$\Psi(\lambda)^{\mathsf{H}} x = \lim_{n\to+\infty} f(\lambda)^{1/2} \Phi_n(\lambda)^{\mathsf{H}} x \in \overline{\mathrm{Im}} f(\lambda)^{1/2} = \mathrm{Im} f(\lambda)^{1/2} \; .$$

Hence $\mathrm{Im}\Psi(\lambda)^{\mathsf{H}} \subset \mathrm{Im} f(\lambda)^{1/2} \subset \mathcal{D}\Big(f(\lambda)^{1/2}\Big)^{\dagger}$ where $\Big(f(\lambda)^{1/2}\Big)^{\dagger}$ is the generalized inverse of $f(\lambda)^{1/2}$ (see Appendix A.3). This means that we can define

$$\Theta(\lambda) := \Big(f(\lambda)^{1/2}\Big)^{\dagger} \Psi(\lambda)^{\mathsf{H}} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{H}_0) \; .$$

Defining $\Theta(\lambda) = 0$ for $\lambda \in B^c$, we get that $\Theta \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{H}_0)$. This implies that the function $\Phi : \lambda \mapsto \Theta(\lambda)^{\mathsf{H}}$ is in $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ and we have

$$\begin{aligned}
\int \left\| \Phi(\lambda) f(\lambda)^{1/2} \right\|_2^2 \|\nu\|_1(\mathrm{d}\lambda) &= \int \left\| f(\lambda)^{1/2} \Theta(\lambda) \right\|_2^2 \|\nu\|_1(\mathrm{d}\lambda) \\
&= \int_A \left\| f(\lambda)^{1/2} \left(f(\lambda)^{1/2}\right)^{\dagger} \Psi(\lambda) \right\|_2^2 \|\nu\|_1(\mathrm{d}\lambda) \\
&= \int_A \left\| P_{\mathrm{Im} f(\lambda)^{1/2}} \Psi(\lambda) \right\|_2^2 \|\nu\|_1(\mathrm{d}\lambda) \\
&= \int_A \|\Psi(\lambda)\|_2^2 \|\nu\|_1(\mathrm{d}\lambda) \\
&< +\infty \; .
\end{aligned}$$

Hence $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Finally

$$\begin{aligned}
\|\Phi - \Phi_n\|^2_{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} &= \int \left\| \Phi f^{1/2} - \Phi_n f^{1/2} \right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \int \left\| f^{1/2}\Theta - f^{1/2} \Phi_n^{\mathsf{H}} \right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \int_A \left\| \Psi^{\mathsf{H}} - \Psi_n^{\mathsf{H}} \right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \int_A \|\Psi - \Psi_n\|_2^2 \, \mathrm{d}\|\nu\|_1 \\
&\xrightarrow[n\to+\infty]{} 0
\end{aligned}$$

that is, $(\Phi_n)_{n\in\mathbb{N}}$ converges to Φ in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ thus concluding the proof of completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. ☐

C Locally compact Abelian groups

A topological group is a group $(\mathbb{G},+)$ (with null element 0) endowed with a topology for which the addition and the inversion maps are continuous in $\mathbb{G}\times\mathbb{G}$ and $\mathbb{G}$ respectively. If $\mathbb{G}$ is Abelian (*i.e.* commutative) and is locally compact, Hausdorff for its topology, then it is called a Locally compact Abelian (l.c.a.) group. A character χ of $\mathbb{G}$ is a group homomorphism from $\mathbb{G}$ to the unit circle group $\mathbb{U} := \{z\in\mathbb{C} : |z|=1\}$ that is $\chi:\mathbb{G}\to\mathbb{U}$ and for all $s,t\in\mathbb{G}$, $\chi(s+t)=\chi(s)\chi(t)$. The dual group $\hat{\mathbb{G}}$ of an l.c.a. group $\mathbb{G}$ is the set of continuous characters of $\mathbb{G}$. In particular, $\chi(0)=1$ and $\overline{\chi(t)}=\chi(t)^{-1}=\chi(-t)$ for all $t\in\mathbb{G}$. $\hat{\mathbb{G}}$ is a multiplicative Abelian group if we define the product of $\chi_1,\chi_2\in\hat{\mathbb{G}}$, as $\chi_1\chi_2 : t\mapsto\chi_1(t)\chi_2(t)$, the identity element as $\hat{e} : t\mapsto 1$ and the inverse of $\chi\in\hat{\mathbb{G}}$ as $\chi^{-1} : t\mapsto\chi(t)^{-1}=\overline{\chi(t)}$. $\hat{\mathbb{G}}$ becomes an l.c.a. group when endowed with the compact-open topology, that is the topology for which $\chi_n\to\chi$ in $\hat{\mathbb{G}}$ if and only if for every compact $K\subset\mathbb{G}$, $\chi_n\to\chi$ uniformly on K *i.e.* $\sup_{t\in K}|\chi_n(t)-\chi(t)| \xrightarrow[n\to+\infty]{} 0$.

A result known as the Pontryagin Duality Theorem (see [57, Theorem 1.7.2]) states that $\mathbb{G}$ and $\hat{\hat{\mathbb{G}}}$ are isomorphic via the evaluation map $\begin{array}{ccc}\mathbb{G} & \to & \hat{\hat{\mathbb{G}}}\\ t & \mapsto & \mathrm{e}_t\end{array}$ where $\mathrm{e}_t : \chi\mapsto\chi(t)$ in the sense that this map is a bijective continuous homomorphisms with continuous inverse. In particular, this means that $\{\mathrm{e}_t : t\in\mathbb{G}\}$ is the set of characters of $\hat{\mathbb{G}}$ (*i.e.* $\hat{\hat{\mathbb{G}}}$). The following theorem will be very useful.

Theorem C.1. *Let $\mathbb{G}$ be an l.c.a. group and μ a regular finite non-negative measure on $(\mathbb{G},\mathcal{B}(\mathbb{G}))$. Then for all Banach space E,*

$$L^2(\mathbb{G},\mathcal{B}(\mathbb{G}),E,\mu) = \overline{\mathrm{Span}}^{L^2(\mathbb{G}\mathcal{B}(\mathbb{G},E,\mu)}\left(t\mapsto\chi(t)x \,:\, \chi\in\hat{\mathbb{G}}\,,\ x\in E\right)\,.$$

By duality, we get that, for all regular finite non-negative measure μ on $(\mathbb{G},\mathcal{B}(\mathbb{G}))$ and Banach space E,

$$L^2(\hat{\mathbb{G}},\mathcal{B}(\hat{\mathbb{G}}),E,\mu) = \overline{\mathrm{Span}}^{L^2(\mathbb{G},\mathcal{B}(\mathbb{G}),E,\mu)}\left(\chi\mapsto\chi(t)x \,:\, t\in\mathbb{G}\,,\ x\in E\right)\,.$$

Proof. The space $\mathrm{Span}\left(\hat{\mathbb{G}}\right)$ satisfies the conditions of the Stone-Weierstrass theorem (see [14]) and therefore is uniformly dense in $\mathcal{C}_0(\mathbb{G})\supset\mathcal{C}_c(\mathbb{G})$. This implies that $\mathrm{Span}\left(t\mapsto\chi(t)x \,:\, \chi\in\hat{\mathbb{G}}, x\in E\right)$ is uniformly dense in $\mathrm{Span}\left(t\mapsto f(t)x \,:\, f\in\mathcal{C}_c(\mathbb{G}), x\in E\right)$ which is itself uniformly dense in $\mathcal{C}_c(\mathbb{G},E)$ by [63, Proposition 44.2]. Since μ is finite, uniform density implies density in L^2-norm and therefore we have shown that $\mathrm{Span}\left(t\mapsto\chi(t)x \,:\, \chi\in\hat{\mathbb{G}}, x\in E\right)$ is dense in $\mathcal{C}_c(\mathbb{G},E)$ in L^2-norm. The result follows because, since μ is regular, $\mathcal{C}_c(\mathbb{G},E)$ is dense in $L^2(\mathbb{G},\mathcal{B}(\mathbb{G}),E,\mu)$ for the L^2-norm. □

It is straightforward to verify that $\mathbb{Z}$ is an l.c.a. group for the addition and discrete topology (the open sets are the subsets of $\mathbb{Z}$, in this case every mapping from $\mathbb{Z}$ to any topological space is continuous). Then $\chi\in\hat{\mathbb{Z}}$ if and only if for all $t,s\in\mathbb{Z}$, $\chi(t+s)=\chi(t)\chi(s)$ and therefore $\hat{\mathbb{Z}}=\left\{\begin{array}{ccc}\mathbb{Z} & \to & \mathbb{U}\\ t & \mapsto & z^t\end{array} : z\in\mathbb{U}\right\}$. Since the compact sets of $\mathbb{Z}$ are the finite subsets of $\mathbb{Z}$, the compact-open topology on $\hat{\mathbb{Z}}$ is the same as the one induced by pointwise convergence. It is then easy to show that $\hat{\mathbb{Z}}$, $\mathbb{U}$ and $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$ are isomorphic (from $\hat{\mathbb{Z}}$ to $\mathbb{U}$ take $\chi\mapsto\chi(1)$ and from $\mathbb{T}$ to $\mathbb{U}$ take $\lambda\mapsto\mathrm{e}^{\mathrm{i}\lambda}$). In this case we identify $\hat{\mathbb{Z}}$ and $\mathbb{T}$ which is in general represented by $(-\pi,\pi]$. The other classical example of l.c.a. group is $\mathbb{R}$ for the addition and usual topology. It can be shown that $\hat{\mathbb{R}}=\left\{\begin{array}{ccc}\mathbb{R} & \to & \mathbb{U}\\ t & \mapsto & \mathrm{e}^{\mathrm{i}t\lambda}\end{array} : \lambda\in\mathbb{R}\right\}$ (see for example [12, Theorem 9.11.] where the idea is to show that the fact that $\chi\in\hat{\mathbb{R}}$ satisfies $\chi(t+s)=\chi(t)\chi(s)$ implies that χ must be differentiable and satisfies a first order differential equation leading to the result). Then $\hat{\mathbb{R}}$ and $\mathbb{R}$ are isomorphic via the mapping $\lambda\mapsto(t\mapsto\mathrm{e}^{\mathrm{i}t\lambda})$. In this case we identify $\hat{\mathbb{R}}$ and $\mathbb{R}$.

References

[1] Warren Ambrose. Spectral resolution of groups of unitary operators. *Duke Math. J.*, 11(3):589–595, 09 1944. doi: 10.1215/S0012-7094-44-01151-8. URL https://doi.org/10.1215/S0012-7094-44-01151-8.

[2] Edmond Arnous. Sur les groupes continus de transformations unitaires de l'espace de hilbert: Une extension d'un théorème de m.h. stone. *Commentarii Mathematici Helvetici*, 19(1):50–60, Dec 1946. ISSN 1420-8946. doi: 10.1007/BF02565945. URL https://doi.org/10.1007/BF02565945.

[3] SK Berberian et al. Naimark's moment theorem. *The Michigan Mathematical Journal*, 13(2):171–184, 1966.

[4] Sterling K. Berberian. *Notes on spectral theory*. Van Nostrand Mathematical Studies, No. 5. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[5] István Berkes, Lajos Horváth, and Gregory Rice. On the asymptotic normality of kernel estimators of the long run covariance of functional time series. *Journal of Multivariate Analysis*, 144:150 – 175, 2016. ISSN 0047-259X. doi: https://doi.org/10.1016/j.jmva.2015.11.005. URL http://www.sciencedirect.com/science/article/pii/S0047259X15002730.

[6] D. Bosq. *Linear processes in function spaces*, volume 149 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 2000. ISBN 0-387-95052-4. doi: 10.1007/978-1-4612-1154-9. URL https://doi.org/10.1007/978-1-4612-1154-9. Theory and applications.

[7] Richard C. Bradley. Basic properties of strong mixing conditions. a survey and some open questions. *Probab. Surveys*, 2:107–144, 2005. doi: 10.1214/154957805100000104. URL https://doi.org/10.1214/154957805100000104.

[8] David R. Brillinger. *Time series*, volume 36 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. ISBN 0-89871-501-6. doi: 10.1137/1.9780898719246. URL https://doi.org/10.1137/1.9780898719246. Data analysis and theory, Reprint of the 1981 edition.

[9] Peter J Brockwell and Richard A Davis. *Time Series: Theory and Methods*. Springer-Verlag, Berlin, Heidelberg, 1986. ISBN 0387964061.

[10] Peter J. Brockwell and Richard A. Davis. *Time series: theory and methods*. Springer series in statistics. Springer, New York, 2nd ed edition, 1996. ISBN 978-0-387-97429-3 978-3-540-97429-1.

[11] James K. Brooks. On the vitali-hahn-saks and nikodym theorems. *Proceedings of the National Academy of Sciences of the United States of America*, 64(2):468–471, 1969. ISSN 00278424. URL http://www.jstor.org/stable/59771.

[12] J.B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 1994. ISBN 9780387972459. URL https://books.google.fr/books?id=ix4P1e6AkeIC.

[13] John B. Conway. *A course in operator theory*, volume 21 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000. ISBN 0-8218-2065-6.

[14] Louis de Branges. The stone-weierstrass theorem. *Proceedings of the American Mathematical Society*, 10(5):822–822, May 1959. doi: 10.1090/s0002-9939-1959-0113131-7. URL https://doi.org/10.1090/s0002-9939-1959-0113131-7.

[15] Anne van Delft. A note on quadratic forms of stationary functional time series under mild conditions. *Stochastic Processes and their Applications*, 2019. ISSN 0304-4149. doi: https://doi.org/10.1016/j.spa.2019.12.002. URL http://www.sciencedirect.com/science/article/pii/S0304414919303680.

[16] Anne van Delft and Michael Eichler. A note on herglotz's theorem for time series on function spaces. *Stochastic Processes and their Applications*, 130(6):3687 – 3710, 2020. ISSN 0304-4149. doi: https://doi.org/10.1016/j.spa.2019.10.006. URL http://www.sciencedirect.com/science/article/pii/S030441491830752X.

[17] J. Diestel and J.J. Uhl. *Vector Measures*. Mathematical surveys and monographs. American Mathematical Society, 1977. ISBN 9780821815151. URL https://books.google.fr/books?id=NCm4E2By8DQC.

[18] Nicolae Dinculeanu. *Vector measures*. Pergamon Press, Oxford, 1967.

[19] Nicolae Dinculeanu. *Vector integration and stochastic integration in Banach spaces*, volume 48. John Wiley & Sons, 2011.

[20] P. Doukhan. *Mixing: properties and examples*. Lecture notes in statistics. Springer-Verlag, 1994. ISBN 9780387942148. URL https://books.google.fr/books?id=xgnvAAAAMAAJ.

[21] Paul Doukhan, Pascal Massart, and Emmanuel Rio. The functional central limit theorem for strongly mixing processes. *Annales de l'I.H.P. Probabilités et statistiques*, 30(1):63–82, 1994. URL http://www.numdam.org/item/AIHPB_1994__30_1_63_0.

[22] Frédéric Ferraty and Philippe Vieu. *Nonparametric functional data analysis*. Springer Series in Statistics. Springer, New York, 2006. ISBN 0-387-30369-3; 978-0387-30369-7. Theory and practice.

[23] P.A. Fillmore. *Notes on operator theory*. Van Nostrand Reinhold mathematical studies. Van Nostrand Reinhold Co., 1970. URL https://books.google.fr/books?id=u1mqAAAAIAAJ.

[24] I. Gohberg, S. Goldberg, and M. Kaashoek. *Basic Classes of Linear Operators*. Birkhäuser Basel, 2003. ISBN 9783764369309. URL https://books.google.fr/books?id=aFc_1wnpuLUC.

[25] C. W. Groetsch. *Generalized inverses of linear operators: representation and approximation*. Marcel Dekker, Inc., New York-Basel, 1977. Monographs and Textbooks in Pure and Applied Mathematics, No. 37.

[26] Peter Hall, Hans-Georg Müller, and Jane-Ling Wang. Properties of principal component methods for functional and longitudinal data analysis. *The Annals of Statistics*, 34(3): 1493–1517, 2006. ISSN 00905364. URL http://www.jstor.org/stable/25463465.

[27] R. Holmes. Mathematical foundations of signal processing. *SIAM Review*, 21(3):361–388, 1979. doi: 10.1137/1021053. URL https://doi.org/10.1137/1021053.

[28] Siegfried Hörmann and P. Kokoszka. Weakly dependent functional data. *The annals of statistics*, 38(3):1845–1884, 2010. ISSN 0090-5364. Language of publication: en.

[29] Siegfried Hörmann, Lukasz Kidziński, and Marc Hallin. Dynamic functional principal components. *J. R. Stat. Soc. Ser. B. Stat. Methodol.*, 77(2):319–348, 2015. ISSN 1369-7412. doi: 10.1111/rssb.12076. URL https://doi.org/10.1111/rssb.12076.

[30] L. Horváth and P. Kokoszka. *Inference for Functional Data with Applications*. Springer Series in Statistics. Springer New York, 2012. ISBN 9781461436553. URL https://books.google.fr/books?id=OVezLB__ZpYC.

[31] Lajos Horváth, Piotr Kokoszka, and Ron Reeder. Estimation of the mean of functional time series and a two-sample problem. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 75(1):103–122, 2013. doi: 10.1111/j.1467-9868.2012.01032.x. URL https://rss.onlinelibrary.wiley.com/doi/abs/10.1111/j.1467-9868.2012.01032.x.

[32] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras. Volume I.* Fundamentals of the Theory of Operator Algebras. American Mathematical Society, 1997. ISBN 9780821808191. URL https://books.google.fr/books?id=Q3J6TV6euVYC.

[33] Yûichirô Kakihara. *Multidimensional Second Order Stochastic Processes.* World Scientific, 1997. doi: 10.1142/3348. URL https://www.worldscientific.com/doi/abs/10.1142/3348.

[34] G Kallianpur and V Mandrekar. Spectral theory of stationary h-valued processes. *Journal of Multivariate Analysis*, 1(1):1–16, 1971.

[35] Irving Kaplansky. Modules over operator algebras. *American Journal of Mathematics*, 75(4):839–858, 1953. ISSN 00029327, 10806377. URL http://www.jstor.org/stable/2372552.

[36] J. Klepsch, C. Klüppelberg, and T. Wei. Prediction of functional ARMA processes with an application to traffic data. *Econometrics and Statistics*, 1:128–149, January 2017. ISSN 24523062. doi: 10.1016/j.ecosta.2016.10.009. URL https://linkinghub.elsevier.com/retrieve/pii/S245230621630020X.

[37] Piotr Kokoszka. Dependent functional data. *ISRN Probability and Statistics*, 2012. doi: doi:10.5402/2012/958254.

[38] Piotr Kokoszka and Neda Mohammadi Jouzdani. Frequency domain theory for functional time series: Variance decomposition and an invariance principle. *Bernoulli*, 26(3):2383–2399, 08 2020. doi: 10.3150/20-BEJ1199. URL https://doi.org/10.3150/20-BEJ1199.

[39] Piotr Kokoszka and Matthew Reimherr. Asymptotic normality of the principal components of functional time series. *Stochastic Processes and their Applications*, 123(5):1546–1562, 2013. doi: 10.1016/j.spa.2012.12.011. URL https://ideas.repec.org/a/eee/spapps/v123y2013i5p1546-1562.html.

[40] A. N. Kolmogoroff. Stationary sequences in Hilbert's space. *Bolletin Moskovskogo Gosudarstvenogo Universiteta. Matematika*, 2:40pp, 1941.

[41] SHIGE TOSHI Kuroda. An abstract stationary approach to perturbation of continuous spectra and scattering theory. *Journal d'Analyse Mathématique*, 20(1):57–117, 1967.

[42] Yehua Li and Tailen Hsing. Uniform convergence rates for nonparametric regression and principal component analysis in functional/longitudinal data. *The Annals of Statistics*, 38(6):3321–3351, 2010. ISSN 00905364. URL http://www.jstor.org/stable/29765266.

[43] Weidong Liu and Wei Biao Wu. Asymptotics of spectral density estimates. *Econometric Theory*, 26(4):1218–1245, 2010. doi: 10.1017/S026646660999051X.

[44] L.H. Loomis. *An introduction to abstract harmonic analysis.* University series in higher mathematics. Van Nostrand, 1953. URL https://books.google.fr/books?id=aNg-AAAAIAAJ.

[45] V. Mandrekar and H. Salehi. The square-integrability of operator-valued functions with respect to a non-negative operator-valued measure and the kolmogorov isomorphism theorem. *Indiana University Mathematics Journal*, 20(6):545–563, 1970. ISSN 00222518, 19435258. URL http://www.jstor.org/stable/24890118.

[46] P. Masani. Recent trends in multivariate prediction theory. Technical report, Defense Technical Information Center, Fort Belvoir, VA, January 1966. URL http://www.dtic.mil/docs/citations/AD0630756.

[47] Florence Merlevède. On the central limit theorem and its weak invariance principle for strongly mixing sequences with values in a hilbert space via martingale approximation. *Journal of Theoretical Probability*, 16(3):625–653, Jul 2003. ISSN 1572-9230. doi: 10.1023/A:1025668415566. URL https://doi.org/10.1023/A:1025668415566.

[48] Florence Merlevède, Magda Peligrad, and Sergey Utev. Sharp conditions for the clt of linear processes in a hilbert space. *Journal of Theoretical Probability*, 10:681–693, 1997.

[49] M Neumark. Positive definite operator functions on a commutative group. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 7(5):237–244, 1943.

[50] Victor M. Panaretos and Shahin Tavakoli. Fourier analysis of stationary time series in function space. *Ann. Statist.*, 41(2):568–603, 2013. ISSN 0090-5364. doi: 10.1214/13-AOS1086. URL https://doi.org/10.1214/13-AOS1086.

[51] Victor M. Panaretos and Shahin Tavakoli. Cramer-karhunen-loeve representation and harmonic principal component analysis of functional time series. *Stochastic Processes And Their Applications*, 123(7):29. 2779–2807, 2013.

[52] R. Payen. Fonctions aléatoires du second ordre à valeurs dans un espace de hilbert. *Annales de l'I.H.P. Probabilités et statistiques*, 3(4):323–396, 1967. URL http://www.numdam.org/item/AIHPB_1967__3_4_323_0.

[53] J. Ramsay and B.W. Silverman. *Functional Data Analysis*. Springer Series in Statistics. Springer New York, 2006. ISBN 9780387227511. URL https://books.google.fr/books?id=REzuyz_V6OQC.

[54] Milton Rosenberg et al. The square-integrability of matrix-valued functions with respect to a non-negative hermitian measure. *Duke Mathematical Journal*, 31(2):291–298, 1964.

[55] M. Rosenblatt. Asymptotic normality, strong mixing and spectral density estimates. *Ann. Probab.*, 12(4):1167–1180, 11 1984. doi: 10.1214/aop/1176993146. URL https://doi.org/10.1214/aop/1176993146.

[56] Tomáš Rubín and Victor M. Panaretos. Sparsely observed functional time series: estimation and prediction. *Electronic Journal of Statistics*, 14(1):1137–1210, 2020. ISSN 1935-7524. doi: 10.1214/20-ejs1690. URL http://dx.doi.org/10.1214/20-EJS1690.

[57] W. Rudin. *Fourier Analysis on Groups*. A Wiley-interscience publication. Wiley, 1990. ISBN 9780471523642.

[58] Habib Salehi. Stone's theorem for a group of unitary operators over a hilbert space. *Proceedings of the American Mathematical Society*, 31(2):480–484, 1972. ISSN 00029939, 10886826. URL http://www.jstor.org/stable/2037557.

[59] Xiaofeng Shao and Biao Wu Wei. Asymptotic spectral theory for nonlinear time series. *Annals of Statistics*, 35(4):1773–1801, 8 2007. ISSN 0090-5364. doi: 10.1214/009053606000001479.

[60] Felix Spangenberg. Strictly stationary solutions of ARMA equations in Banach spaces. *Journal of Multivariate Analysis*, 121:127–138, October 2013. ISSN 0047259X. doi: 10.1016/j.jmva.2013.06.007. URL https://linkinghub.elsevier.com/retrieve/pii/S0047259X13001280.

[61] B. Szőkefalvi-Nagy. *Appendix to Functional Analysis: Extensions of Linear Transformations in Hilbert Space which Extend Beyond this Space*. Ungar, 1960. URL https://books.google.fr/books?id=fIjgAAAAMAAJ.

[62] Shahin Tavakoli. *Fourier Analysis of Functional Time Series, with Applications to DNA Dynamics*. PhD thesis, MATHAA, EPFL, 2014.

[63] F. Treves. *Topological Vector Spaces, Distributions and Kernels*. Dover books on mathematics. Academic Press, 2006. ISBN 9780486453521. URL https://books.google.fr/books?id=kClvQ1qk9r8C.

[64] N. Wiener and P. Masani. The prediction theory of multivariate stochastic processes: I. the regularity condition. *Acta Math.*, 98:111–150, 1957. doi: 10.1007/BF02404472. URL https://doi.org/10.1007/BF02404472.

[65] Fang Yao, Hans-Georg Müller, and Jane-Ling Wang. Functional data analysis for sparse longitudinal data. *Journal of the American Statistical Association*, 100(470):577–590, 2005. doi: 10.1198/016214504000001745. URL https://doi.org/10.1198/016214504000001745.