On Minimum Entropy and Gaussian Transport

Entropy 2018: From Physics to Information Sciences and Geometry

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Outline

Introduction
What is Entropy?
Max/Min Entropy Principles
Equivalence to the Entropy Power Inequality
A Proof that Shannon Missed
Generalization to Linear Transformations
Shannon vs. Rényi
A Proof that Shannon Missed (Revisited)
Generalization to Rényi Entropies
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THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist\(^1\) and Hartley\(^2\) on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.
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Generalization to Rényi Entropies
\[ h(X) = \int f(x) \log \frac{1}{f(x)} \, dx \text{ where } X \sim f \]

“You should call it entropy [...] no one really knows what entropy really is, so in a debate you will always have the advantage.”

John von Neumann (1903–1957)
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“\textit{In the continuous case it is convenient to work not with the entropy } H \textit{of an ensemble but with a derived quantity which we will call the entropy power.}” [Shannon’1948]
Entropy-Power

Definition ([Shannon48])

- **Power**: $P(X) = \mathbb{E}(X^2)$
- **Entropy-Power**: $N(X)$

power of a Gaussian $X^*$ having the same entropy
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Since $h(X^*) = \frac{1}{2} \log(2\pi e P(X^*))$:

$$N(X) = \frac{e^{2h(X)}}{2\pi e}$$
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“entropy power”

• Scaling Property:

\[
P(aX) = a^2 P(X) \quad N(aX) = a^2 N(X)
\]
Entropy-Power Inequality (EPI)

for any $X \perp \perp Y$:

$$P(X + Y) = P(X) + P(Y)$$
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$$P(X + Y) = P(X) + P(Y)$$

Theorem (stated by Shannon, 1948)

$$N(X + Y) \geq N(X) + N(Y)$$

with equality iff $X, Y$ are Gaussian.

The following result is derived in Appendix 6.

**Theorem 15:** Let the average power of two ensembles be $N_1$ and $N_2$ and let their entropy powers be $\overline{N}_1$ and $\overline{N}_2$. Then the entropy power of the sum, $\overline{N}_3$, is bounded by

$$\overline{N}_1 + \overline{N}_2 \leq \overline{N}_3 \leq N_1 + N_2.$$
Entropy-Power Inequality (EPI)

for any $X \perp \perp Y$:

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$$\bar{N}_1 + \bar{N}_2 \leq \bar{N}_3 \leq N_1 + N_2.$$
Shannon’s 1948 “proof” (Appendix 6).

A variational argument: $h(X + Y)$ for fixed $h(X)$ and $h(Y)$ has a stationary point when $X, Y$ are Gaussian.

This does not exclude local minima/maxima/saddle points.
The EPI has a Long History

1948  Stated and “proved” by Shannon in his seminal paper
1959  Stam’s proof using Fisher information
1965  Blachman’s exposition of Stam’s proof in IEEE Trans. IT
1978  Lieb’s proof using strengthened Young’s inequality
1991  Dembo-Cover-Thomas’ review of Stam’s & Lieb’s proofs
1991  Carlen-Soffer 1D variation of Stam’s proof
2000  Szarek-Voiculescu variant with Brunn-Minkowski inequality
2006  Guo-Shamai-Verdú proof based on the I-MMSE relation
2007  Rioul’s proof based on Mutual Information
2014  Wang-Madiman strengthening using Rényi entropies
2016  Courtade’s strengthening
2017  Yet another simple proof!
Applications of the EPI:

- nonGaussian Capacity [Shannon’48]
  Gaussian means worst noise / Gaussian means best signal
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- blind deconvolution / source separation [Donoho81]
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Ingredients

random $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \in \mathbb{R}^n$ with independent components $X_i$
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- random \( X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \in \mathbb{R}^n \) with independent components \( X_i \)
- a linear transformation: \( X \mapsto AX \)
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  - $A$ is an $m \times n$ matrix with $m \leq n$
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- $\max h(AX)$ or $\min h(AX)$?
Max/Min Entropy Principle

Let $X^*$ be Gaussian with independent components $X_i^*$ of same variances: $\text{Var}(X_i^*) = \text{Var}(X_i)$.

Theorem (Maximum Entropy Principle)

$$h(AX) \leq h(AX^*) \quad \text{with equality iff } X \text{ is Gaussian}$$
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**Proof:** \( h(AX^*) - h(AX) = D_{KL}(AX \| AX^*) \geq 0 \) \( \square \)
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Theorem (Minimum Entropy Principle)

$$h(AX) \geq h(A\mathbf{X}^*) \quad \text{with equality iff } X \text{ is Gaussian} \ldots$$

... or $A$ is trivial

- Closeness to normality by linear filtering
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- Application to deconvolution / blind separation
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- Application to deconvolution / blind separation
- Proof: involved!
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Generalization to Rényi Entropies
Simplest nontrivial case: \((m, n) = (1, 2)\)

Take \(\mathbf{A} = \begin{pmatrix} a & b \end{pmatrix}\) with nonzero \(a, b\) (nontrivial mixture).

Theorem (MinEnt for \((m, n) = (1, 2)\))

For any two independent \(X, Y\), letting \(X^*, Y^*\) independent Gaussian s.t. \(h(X^*) = h(X), h(Y) = h(Y^*)\),

\[ h(aX + bY) \geq h(aX^* + bY^*) \]

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**Definition (Entropy Power [Shannon’48])**

Entropy Power = Power of a Gaussian noise with the same entropy:

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N(X) = \text{Var}(X^*) \quad \text{where} \quad h(X^*) = h(X)
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i.e., since \(h(X^*) = \frac{1}{2} \log(2\pi e \text{Var}(X^*))\),

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Lemma (inverse function sampling method)

*If U is uniform in \([0, 1]\) and \(X\) has c.d.f. \(F(x) = \mathbb{P}(X \leq x)\), then \(F^{-1}(U)\) has the same distribution as \(X\).*

**Proof.**

\[
\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x).
\]
Ingredients: “Optimal Transport”

Lemma (inverse function sampling method)

If $U$ is uniform in $[0, 1]$ and $X$ has c.d.f. $F(x) = \mathbb{P}(X \leq x)$, then $F^{-1}(U)$ has the same distribution as $X$.

Corollary (monotonic increasing transport $T = F^{-1} \circ G$)

Let $F, G$ be two c.d.f’s. Then $X^* \sim G \implies X = T(X^*) \sim F$.

Proof.

$U = G(X^*) \sim \text{uniform}; \quad T(X^*) = F^{-1}(G(X^*)) = F^{-1}(U) \sim F$. $\square$
**Ingredients: “Optimal Transport”**

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- \( nD \) generalization: Knöthe map, Brenier map...
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- $nD$ generalization: Knöthe map, Brenier map...
- Used in optimal transport theory
Ingredients: “Optimal Transport”

Lemma (Change of variable [Shannon’48])

For any continuous $X, X^*$, monotonic increasing transport $T(X^*) \sim X$,

\[ h(X) = h(T(X^*)) = h(X^*) + \mathbb{E} \log T'(X^*) \]
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For any continuous $X, X^*$, monotonic increasing transport $T(X^*) \sim X$,

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h(X) = h(T(X^*)) = h(X^*) + \mathbb{E} \log T'(X^*)
\]

Proof.

Proof: make the change of variable $x = T(x^*)$ in

\[
h(X) = \int f_X(x) \log \frac{1}{f_X(x)} \, dx = \int f_X(T(x^*)) T'(x^*) \log \frac{1}{f_X(T(x^*))} \, dx^*
\]
Ingredients: “Optimal Transport”

Lemma (Change of variable [Shannon’48])

For any continuous $X, X^*$, monotonic increasing transport $T(X^*) \sim X$,

$$h(X) = h(T(X^*)) = h(X^*) + \mathbb{E} \log T'(X^*)$$

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- in particular $h(aX) = h(X) + \log |a| \iff N(aX) = a^2 N(X)$;
Ingredients: “Optimal Transport”

**Lemma (Change of variable [Shannon’48])**

For any continuous $X, X^*$, monotonic increasing transport $T(X^*) \sim X$,

$$h(X) = h(T(X^*)) = h(X^*) + E \log T'(X^*)$$

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- in particular $h(aX) = h(X) + \log |a| \iff N(aX) = a^2 N(X)$;
- more generally in $nD$: $h(T(X^*)) = h(X^*) + E \log |\det T'(X^*)|$. 

On Minimum Entropy and Gaussian Transport
A Proof that Shannon Missed

Proceed to prove the inequality

\[ h(aX + bY) \geq h(aX^* + bY^*) \]

where \( X^*, Y^* \) are indep. Gaussian s.t. \( h(X^*) = h(X), h(Y) = h(Y^*) \)
A Proof that Shannon Missed

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1. We may assume \( h(X) = h(Y) \).
A Proof that Shannon Missed

Proceed to prove the inequality $\text{h}(aX + bY) \geq \text{h}(aX^* + bY^*)$ where $X^*, Y^*$ are indep. Gaussian s.t. $\text{h}(X^*) = \text{h}(X) = \text{h}(Y) = \text{h}(Y^*)$

1. We may assume $\text{h}(X) = \text{h}(Y)$. Otherwise:
   - set $c = e^{-\text{h}(X)}$ and $d = e^{-\text{h}(Y)}$ so that $\text{h}(cX) = \text{h}(dY)$;
   - apply the above to $cX$ and $dY$. 

2. We may always normalize: $a^2 + b^2 = 1$. Otherwise:
   - divide $a, b$ by $\Delta = \sqrt{a^2 + b^2}$;
   - the log $\Delta$ terms cancel.

3. Make the changes of variables $X = T(X^*)$, $Y = U(Y^*)$:
   One is led to prove $\text{h}(aT(X^*) + bU(Y^*)) \geq \text{h}(aX^* + bY^*)$

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   $\tilde{Y} = -bX^* + aY^*$ so that $\tilde{X}, \tilde{Y}$ are i.i.d. Gaussian and $X^* = a\tilde{X} - b\tilde{Y}$, $Y^* = b\tilde{X} + a\tilde{Y}$. 

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Olivier Rioul

On Minimum Entropy and Gaussian Transport
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One is led to prove $h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*)$.

$\tilde{X}, \tilde{Y}$ are i.i.d. Gaussian and $X^* = a\tilde{X} - b\tilde{Y}$, $Y^* = b\tilde{X} + a\tilde{Y}$. 
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One is led to prove

\[ h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \]

\(\tilde{X}, \tilde{Y}\) are i.i.d. Gaussian and \(X^* = a\tilde{X} - b\tilde{Y}\), \(Y^* = b\tilde{X} + a\tilde{Y}\).

5. Since conditioning reduces entropy:

\[
h(aT(X^*) + bU(Y^*)) = h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y})) \\
\geq h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}) | \tilde{Y})
\]
A Proof that Shannon Missed

One is led to prove

\[ h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \]

\( \tilde{X}, \tilde{Y} \) are i.i.d. Gaussian and

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\begin{align*}
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  &\geq h( aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}) | \tilde{Y}) \\
  &\geq h( aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}) | \tilde{Y}) \\
  &\geq h( aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}) | \tilde{X})
\end{align*}
\]
A Proof that Shannon Missed

One is led to prove

\[ h(aT(X^*) + bU(Y^*)) \geq h(a\tilde{X} + b\tilde{Y}) \]

\( \tilde{X}, \tilde{Y} \) are i.i.d. Gaussian and $X^* = a\tilde{X} - b\tilde{Y}$, $Y^* = b\tilde{X} + a\tilde{Y}$.

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\end{align*}
\]

6. By the change of variable:

\[
\begin{align*}
    &\quad = h(\tilde{X} | \tilde{Y}) + \mathbb{E} \log T'_Y(\tilde{X})
\end{align*}
\]
A Proof that Shannon Missed

One is led to prove \( h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \)

\( \tilde{X}, \tilde{Y} \) are i.i.d. Gaussian and \( X^* = a\tilde{X} - b\tilde{Y}, \ Y^* = b\tilde{X} + a\tilde{Y} \).

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6. By the change of variable:
\[
= h(\tilde{X}) + \mathbb{E} \log T'_{\tilde{Y}}(\tilde{X})
\]
A Proof that Shannon Missed

One is led to prove

\[ h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \]

\( \tilde{X}, \tilde{Y} \) are i.i.d. Gaussian and \( X^* = a\tilde{X} - b\tilde{Y}, \ Y^* = b\tilde{X} + a\tilde{Y} \).

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h(aT(X^*) + bU(Y^*)) = h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}))
\geq h(\underbrace{aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y})}_{T_{\tilde{Y}}(\tilde{X})} | \tilde{Y})
\]

6. By the change of variable:

\[
= h(\tilde{X}) + \mathbb{E} \log T'_{\tilde{Y}}(\tilde{X})
= h(\tilde{X}) + \mathbb{E} \log(a^2T'(a\tilde{X} - b\tilde{Y}) + b^2U'(b\tilde{X} + a\tilde{Y}))
\]
A Proof that Shannon Missed

One is led to prove \( h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \)

\( X^*, Y^* \) are i.i.d. Gaussian and \( X^* = a\tilde{X} - b\tilde{Y}, \ Y^* = b\tilde{X} + a\tilde{Y} \).

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\begin{align*}
&h(aT(X^*) + bU(Y^*)) = h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y})) \\
&\geq h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}) | \tilde{Y})
\end{align*}
\]

6. By the change of variable: \( T_{\tilde{Y}}(\tilde{X}) \)

\[
\begin{align*}
&= h(\tilde{X}) + E \log T'_{\tilde{Y}}(\tilde{X}) \\
&= h(\tilde{X}) + E \log(a^2T'(a\tilde{X} - b\tilde{Y}) + b^2U'(b\tilde{X} + a\tilde{Y})) \\
&= h(aX^* + bY^*) + E \log(a^2T'(X^*) + b^2U'(Y^*))
\end{align*}
\]
A Proof that Shannon Missed

One is led to prove

\[ h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \]

\( \tilde{X}, \tilde{Y} \) are i.i.d. Gaussian and \( X^* = a\tilde{X} - b\tilde{Y}, \ Y^* = b\tilde{X} + a\tilde{Y} \).

5. Since conditioning reduces entropy:

\[ h(aT(X^*) + bU(Y^*)) = h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y})) \]

\[ \geq h( aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y}) \mid \tilde{Y} ) \]

6. By the change of variable:

\[ T_{\tilde{Y}}(\tilde{X}) \]

\[ = h(\tilde{X}) + \mathbb{E} \log T'_{\tilde{Y}}(\tilde{X}) \]

\[ = h(\tilde{X}) + \mathbb{E} \log (a^2 T'(a\tilde{X} - b\tilde{Y}) + b^2 U'(b\tilde{X} + a\tilde{Y})) \]

\[ = h(aX^* + bY^*) + \mathbb{E} \log (a^2 T'(X^*) + b^2 U'(Y^*)) \]

7. By concavity of the log:

\[ \geq h(aX^* + bY^*) + a^2 \mathbb{E} \log T'(X^*) + b^2 \mathbb{E} \log U'(Y^*) \]
A Proof that Shannon Missed

One is led to prove \[ h(aT(X^*) + bU(Y^*)) \geq h(aX^* + bY^*) \]

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6. By the change of variable:
\[ = h(\tilde{X}) + \mathbb{E} \log T'_{\tilde{Y}}(\tilde{X}) \]
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7. By concavity of the log:
\[ \geq h(aX^* + bY^*) + a^2 \mathbb{E} \log T'(X^*) + b^2 \mathbb{E} \log U'(Y^*) \]
\[ \geq h(X) - h(X^*) = 0 \quad h(Y) - h(Y^*) = 0 \]
A Proof that Shannon Missed

One is led to prove

\[ h(a T(X^*) + b U(Y^*)) \geq h(aX^* + bY^*) \]

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h(a T(X^*) + b U(Y^*)) = h(a T(a\tilde{X} - b\tilde{Y}) + b U(b\tilde{X} + a\tilde{Y}))\]

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\[\geq h(aX^* + bY^*) + a^2 \mathbb{E} \log T'(X^*) + b^2 \mathbb{E} \log U'(Y^*)\]

\[\geq h(aX^* + bY^*) \quad \square\]
Equality Case

For nonzero $a, b$:

- in log concavity inequality:

\[ \mathbb{E} \log(a^2 T'(X^*) + b^2 U'(Y^*)) = a^2 \mathbb{E} \log T'(X^*) + b^2 \mathbb{E} \log U'(Y^*) \]

\[ \implies T'(X^*) = U'(X^*) = c > 0 \text{ constant a.e.} \]
Equality Case

For nonzero $a, b$:

- in log concavity inequality:

$$
E \log(a^2 T'(X^*) + b^2 U'(Y^*)) = a^2 E \log T'(X^*) + b^2 E \log U'(Y^*)
$$

$$
\implies T'(X^*) = U'(X^*) = c > 0 \text{ constant a.e.}
$$

$$
\implies T, U \text{ are linear: } X = T(X^*) = cX^*, \ Y = U(Y^*) = cY^* \text{ Gaussian.}
$$
Equality Case

For nonzero $a, b$:

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\mathbb{E} \log(a^2 T'(X^*) + b^2 U'(Y^*)) = a^2 \mathbb{E} \log T'(X^*) + b^2 \mathbb{E} \log U'(Y^*)
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\[
\implies c = 1 \text{ since } h(X) = h(X^*), \ h(Y) = h(Y^*).
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Equality Case

For nonzero $a, b$:

- in log concavity inequality:
  \[
  \mathbb{E} \log \left( a^2 T'(X^*) + b^2 U'(Y^*) \right) = a^2 \mathbb{E} \log T'(X^*) + b^2 \mathbb{E} \log U'(Y^*)
  \]
  \[
  \implies T'(X^*) = U'(X^*) = c > 0 \text{ constant a.e.}
  \]
  \[
  \implies T, U \text{ are linear: } X = T(X^*) = cX^*, \ Y = U(Y^*) = cY^* \text{ Gaussian.}
  \]
  \[
  \implies c = 1 \text{ since } h(X) = h(X^*), h(Y) = h(Y^*).
  \]

- in information inequality:
  \[
  h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y})) = h(aT(a\tilde{X} - b\tilde{Y}) + bU(b\tilde{X} + a\tilde{Y})|\tilde{Y})
  \]
  comes for free since $a(a\tilde{X} - b\tilde{Y}) + b(b\tilde{X} + a\tilde{Y}) = \tilde{X}$ is indep of $\tilde{Y}$.  

Outline

Introduction
What is Entropy?
Max/Min Entropy Principles
Equivalence to the Entropy Power Inequality
A Proof that Shannon Missed
Generalization to Linear Transformations
Shannon vs. Rényi
A Proof that Shannon Missed (Revisited)
Generalization to Rényi Entropies
Generalization to Linear Transformations

Proceed to prove $h(AX) \geq h(AX^*)$. 

Generalization to Linear Transformations

Proceed to prove $h(AX) \geq h(AX^*)$.

- We may assume all $X_i$ have the same entropy: Otherwise, introduce $c_i = e^{-h(X_i)}$ and apply the result to the $c_iX_i$. 

Since $h(X_i^*) = h(X_i)$, all $X_i^*$ have the same variance, hence are i.i.d. 

We may assume that $A$ has rank $m \leq n$: Otherwise, the result is trivial: $h(AX) = h(AX^*) = -\infty$.

The difference $h(AX) - h(AX^*)$ is invariant by elementary row operations. By the Gram-Schmidt procedure, we may assume that the rows of $A$ are orthonormal: $AA^t = I$.

Extend $A$ to an orthogonal matrix $A' = (A A_c)$. 

On Minimum Entropy and Gaussian Transport
Generalization to Linear Transformations

Proceed to prove \( h(AX) \geq h(AX^*) \).

- We may assume all \( X_i \) have the same entropy: Otherwise, introduce \( c_i = e^{-h(X_i)} \) and apply the result to the \( c_iX_i \).
- Since \( h(X_i^*) = h(X_i) \), all \( X_i^* \) have the same variance, hence are i.i.d.
Generalization to Linear Transformations

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- The difference $h(AX) - h(AX^*)$ is invariant by elementary row operations. By the Gram-Schmidt procedure, we may assume that the rows of $A$ are orthonormal: $AA^t = I$. 

Generalization to Linear Transformations

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- Extend $A$ to an orthogonal matrix $A' = \begin{pmatrix} A & A_c \end{pmatrix}$
Then let $\tilde{X} = AX^*$ et $\tilde{X}^c = A^c X^*$ so that $\tilde{X}' = \begin{pmatrix} \tilde{X} \\ \tilde{X}^c \end{pmatrix} = A'X^*$ has i.i.d. components. Inverting yields $X^* = A'^t \tilde{X}'$. 
Generalization to Linear Transformations

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By the changes of variables $X_i = T_i(X_i^*)$, since conditioning reduces entropy:

$$h(AX) = h(AT(X^*))$$
$$= h(AT(A'^t \tilde{X}'))$$
$$\geq h(AT(A'^t \tilde{X}' \mid \tilde{X}^c))$$
Generalization to Linear Transformations

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By the changes of variables $X_i = T_i(X_i^*)$, since conditioning reduces entropy:

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$$= h(AT(A'^t \tilde{X}'))$$
$$\geq h(AT(A'^t \tilde{X}') | \tilde{X}^c)$$

But the Jacobian matrix of

$$T_{\tilde{X}^c}(\tilde{X}) = AT(A'^t \tilde{X}') = AT(A'^t \tilde{X} + (A^c)^t \tilde{X}^c)$$

for fixed $\tilde{X}^c$ is

$$T'_{\tilde{X}^c}(\tilde{X}) = AT'(A'^t \tilde{X}')A^t = AT'(X^*)A^t$$

where $T'(X^*) = \text{diag}(T'_i(X_i^*))$.
Generalization to Linear Transformations

The change of variables in the entropy yields

\[ h(AX) \geq h(\mathbf{A} \mathbf{T}(A'^t \tilde{X}') | \tilde{X}^c) \]

\[ = h(\tilde{X}|\tilde{X}^c) + \mathbb{E} \log \det(\mathbf{A} \mathbf{T}'(X^*) \mathbf{A}^t) \]
Generalization to Linear Transformations

The change of variables in the entropy yields

\[ h(AX) \geq h(AT(A^t \tilde{\mathbf{X}}) | \tilde{\mathbf{X}}^c) \]

\[ = h(\tilde{\mathbf{X}}|\tilde{\mathbf{X}}^c) + \mathbb{E} \log \det(AT'(X^*)A^t) \]

By the concavity of the logarithm:

\[ \log \det(AT'(X^*)A^t) \geq \text{tr}(A \cdot \log T'(X^*) \cdot A^t) \]

thus

\[ h(AX) \geq h(\tilde{\mathbf{X}}|\tilde{\mathbf{X}}^c) + \text{tr}(A \cdot \mathbb{E} \log T'(\tilde{\mathbf{X}}) \cdot A^t) \]
Generalization to Linear Transformations

- The change of variables in the entropy yields
  \[ h(A\mathbf{X}) \geq h(\mathbf{A} T(\mathbf{A}' \tilde{\mathbf{X}}') | \tilde{\mathbf{X}}^c) \]
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  thus
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- But \( h(\tilde{\mathbf{X}}|\tilde{\mathbf{X}}^c) = h(\tilde{\mathbf{X}}) = h(A\mathbf{X}^*) \) and
  \[ \mathbb{E} \log T'_i(\tilde{X}_i) = h(T_i(\tilde{X}_i)) - h(\tilde{X}_i) = h(X_i) - h(\tilde{X}_i) = 0; \]
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Generalization to Linear Transformations

The change of variables in the entropy yields

$$h(AX) \geq h(AT'(A^t \tilde{X}') | \tilde{X}^c)$$

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$$h(AX) \geq h(\tilde{X}|\tilde{X}^c) + \text{tr}(A \cdot \mathbb{E} \log T'(\tilde{X}) \cdot A^t)$$

But $$h(\tilde{X}|\tilde{X}^c) = h(\tilde{X}) = h(AX^*)$$ and

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so

$$h(AX) \geq h(AX^*)$$

Equality iff either $$A$$ is trivial or $$T'_i(X_i) = \text{Cst.},$$ hence $$X$$ is Gaussian.
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Shannon’s and Rényi’s Entropies

\[ h(X) = \int f \log \frac{1}{f} \]
Shannon’s and Rényi’s Entropies

\[ h(X) = \int f \log \frac{1}{f} = h_1(X) \]

\[ h_r(X) = \frac{1}{1 - r} \log \int f^r \]

“A mathematician is a device for turning coffee into theorems”

Alfred Renyi (1921–1970)
Lieb’s Restatement of the EPI

Theorem (Lieb, 1978)

\[ e^{2h(X+Y)} \geq e^{2h(X)} + e^{2h(Y)} \]

\[ \iff \text{for any } 0 < \lambda < 1 \]

\[ h(\sqrt{\lambda X} + \sqrt{1 - \lambda Y}) \geq \lambda h(X) + (1 - \lambda)h(Y) \]

Proof.

\[ \implies X = \sqrt{\lambda} X', \ Y = \sqrt{1 - \lambda} Y', \text{ take the log (concavity of the log)} \]

\[ \iff X = X'/\sqrt{\lambda}, \ Y = Y'/\sqrt{1 - \lambda}, \text{ take the exp, assuming } \lambda \text{ such that } h(X) = h(Y), \text{ r.h.s. is } (e^{2h(X)})^\lambda (e^{2h(Y)})^{1-\lambda} = \lambda e^{2h(X)} + (1 - \lambda)e^{2h(Y)}. \]
Lieb’s Restatement of the EPI

Theorem (A generalization:)

\[ e^{2h_r(X+Y)} \geq c \cdot (e^{2h_r(X)} + e^{2h_r(Y)}) \]

\[ \iff \text{for any } 0 < \lambda < 1 \]

\[ h_r(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda h_r(X) + (1-\lambda)h_r(Y) + \log \sqrt{c} \]

Same proof!

\[ \implies X = \sqrt{\lambda}X', \ Y = \sqrt{1-\lambda}Y', \text{ take the log (concavity of the log) } \]

\[ \iff X = X'/\sqrt{\lambda}, \ Y = Y'/\sqrt{1-\lambda}, \text{ take the exp, assuming } \lambda \text{ such that } h_r(X) = h_r(Y), \text{ r.h.s. } (e^{2h_r(X)})^\lambda (e^{2h_r(Y)})^{1-\lambda} = \lambda e^{2h_r(X)} + (1-\lambda)e^{2h_r(Y)} \]
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\[ h_r(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda h_r(X) + (1-\lambda)h_r(Y) + n \log \sqrt{c} \]

Same proof! (with \(c\) independent of the dimension).

\[ \implies X = \sqrt{\lambda}X', Y = \sqrt{1-\lambda}Y', \text{ take the log (concavity of the log)} \]

\[ \iff X = X'/\sqrt{\lambda}, Y = Y'/\sqrt{1-\lambda}, \text{ take the exp, assuming } \lambda \text{ such that } h_r(X) = h_r(Y), \text{ r.h.s. } (e^{2h_r(X)})^\lambda (e^{2h_r(Y)})^{1-\lambda} = \lambda e^{2h_r(X)} + (1-\lambda)e^{2h_r(Y)} \]
Restatement for More than Two Variables

$N$ independent variables $X_1, X_2, \ldots, X_N$.

**Theorem**

\[
e^{2h_r(\sum_i X_i)} \geq c \cdot \sum_i e^{2h_r(X_i)}
\]

$\iff$ for any convex combination ($\sum \lambda_i = 1$)

\[
h_r(\sum_i \lambda_i X_i) \geq \sum_i \lambda_i h_r(X_i) + \frac{n}{2} \log c
\]

Same proof.
Variation for More than Two Variables

$N$ independent variables $X_1, X_2, \ldots, X_N$.

**Theorem**

\[
e^{2\alpha h_r(\sum_i X_i)} \geq \sum_i e^{2\alpha h_r(X_i)}
\]

\[\iff\]

for any convex combination ($\sum_i \lambda_i = 1$)

\[
h_r\left(\sum_i \sqrt{\lambda_i} X_i\right) \geq \sum_i \lambda_i h_r(X_i) + \frac{n}{2} (1/\alpha - 1) H(\lambda)
\]

Same proof.
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A Proof that Shannon Missed (Revisited)
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The Proof that Shannon Missed (Again)

Take any $X \perp Y$ and $X^*, Y^*$ i.i.d. Gaussian. Set $X = T(X^*)$ and $Y = U(Y^*)$. Then

$$h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) - \lambda h(X) - (1 - \lambda) h(Y)$$

$$= h(\sqrt{\lambda}T(X^*) + \sqrt{1-\lambda}U(Y^*)) - \lambda h(T(X^*)) - (1 - \lambda) h(U(Y^*))$$
The Proof that Shannon Missed (Again)

Take any $X \perp Y$ and $X^*, Y^*$ i.i.d. Gaussian. Set $X = T(X^*)$ and $Y = U(Y^*)$. Then

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$$= h(\sqrt{\lambda}T(X^*) + \sqrt{1 - \lambda}U(Y^*)) - \lambda h(T(X^*)) - (1 - \lambda) h(U(Y^*))$$

Compare this to

$$h(\sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^*) - \lambda h(X^*) + (1 - \lambda) h(Y^*)$$

where $\tilde{X}$ is a function of $\tilde{X}^*$ and $\tilde{Y}$ is a function of $\tilde{Y}^*$.
The Proof that Shannon Missed (Again)

Take any $X \perp \perp Y$ and $X^*, Y^*$ i.i.d. Gaussian. Set $X = T(X^*)$ and $Y = U(Y^*)$. Then

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Compare this to

$$h(\sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^*) - \lambda h(X^*) + (1 - \lambda)h(Y^*)$$

$$\left\{\begin{array}{c}
\tilde{X} = \sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^*\\
\tilde{Y} = -\sqrt{1 - \lambda}X^* + \sqrt{\lambda}Y^*
\end{array}\right. \quad \left\{\begin{array}{c}
X^* = \sqrt{\lambda}\tilde{X} - \sqrt{1 - \lambda}\tilde{Y}\\
Y^* = \sqrt{1 - \lambda}\tilde{X} + \sqrt{\lambda}\tilde{Y}
\end{array}\right.$$
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Take any $X \perp \perp Y$ and $X^*, Y^*$ i.i.d. Gaussian. Set $X = T(X^*)$ and $Y = U(Y^*)$. Then

$$h(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) - \lambda h(X) - (1 - \lambda) h(Y)$$

$$= h(\sqrt{\lambda}T(X^*) + \sqrt{1 - \lambda}U(Y^*)) - \lambda h(T(X^*)) - (1 - \lambda) h(U(Y^*))$$

Compare this to

$$h(\sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^*) - \lambda h(X^*) + (1 - \lambda) h(Y^*)$$

$$\begin{cases}
\tilde{X} = \sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^* \\
\tilde{Y} = -\sqrt{1 - \lambda}X^* + \sqrt{\lambda}Y^*
\end{cases} \begin{cases}
X^* = \sqrt{\lambda}\tilde{X} - \sqrt{1 - \lambda}\tilde{Y} \\
Y^* = \sqrt{1 - \lambda}\tilde{X} + \sqrt{\lambda}\tilde{Y}
\end{cases}$$

Then $\sqrt{\lambda}T(X^*) + \sqrt{1 - \lambda}U(Y^*)$ becomes a function of $\tilde{X}, \tilde{Y}$...
A Proof that Shannon Missed (Cont’d)

\[ h(\sqrt{\lambda X} + \sqrt{1 - \lambda Y}) = h(\sqrt{\lambda T(X^*)} + \sqrt{1 - \lambda U(Y^*)}) \]
\[ = h(\sqrt{\lambda T(\sqrt{\lambda \tilde{X}} - \sqrt{1 - \lambda \tilde{Y}})} + \sqrt{1 - \lambda U(\sqrt{1 - \lambda \tilde{X}} + \sqrt{\lambda \tilde{Y}})}) \]
\[ \geq h(\sqrt{\lambda T(\sqrt{\lambda \tilde{X}} - \sqrt{1 - \lambda \tilde{Y}})} + \sqrt{1 - \lambda U(\sqrt{1 - \lambda \tilde{X}} + \sqrt{\lambda \tilde{Y})})|\tilde{Y}) \]
\[ = h(\tilde{X}|\tilde{Y}) + E \log(\lambda T'(\sqrt{\lambda \tilde{X}} - \sqrt{1 - \lambda \tilde{Y}}) + (1 - \lambda)U'(\sqrt{1 - \lambda \tilde{X}} + \sqrt{\lambda \tilde{Y}})) \]
\[ = h(\tilde{X}) + E \log(\lambda T'(X^*) + (1 - \lambda)U'(Y^*)) \]
\[ \geq h(\sqrt{\lambda X^*} + \sqrt{1 - \lambda Y^*}) + \lambda E \log T'(X^*) + (1 - \lambda)E \log U'(Y^*) \]
A Proof that Shannon Missed (Cont’d)

\[
\begin{align*}
    h(\sqrt{\lambda X} + \sqrt{1 - \lambda Y}) &= h(\sqrt{\lambda T(X^*)} + \sqrt{1 - \lambda U(Y^*)}) \\
    &= h(\sqrt{\lambda T(\sqrt{\lambda \tilde{X}} - \sqrt{1 - \lambda \tilde{Y}}) + \sqrt{1 - \lambda U(\sqrt{1 - \lambda \tilde{X}} + \sqrt{\lambda \tilde{Y}})}) \\
    &\geq h(\sqrt{\lambda T(\sqrt{\lambda \tilde{X}} - \sqrt{1 - \lambda \tilde{Y}}) + \sqrt{1 - \lambda U(\sqrt{1 - \lambda \tilde{X}} + \sqrt{\lambda \tilde{Y}})|\tilde{Y}}) \\
    &= h(\tilde{X}|\tilde{Y}) + \mathbb{E} \log(\lambda T'(\sqrt{\lambda \tilde{X}} - \sqrt{1 - \lambda \tilde{Y}}) + (1 - \lambda)U'(\sqrt{1 - \lambda \tilde{X}} + \sqrt{\lambda \tilde{Y}})) \\
    &= h(\tilde{X}) + \mathbb{E} \log(\lambda T'(X^*) + (1 - \lambda)U'(Y^*)) \\
    &\geq h(\sqrt{\lambda X^*} + \sqrt{1 - \lambda Y^*}) + \lambda \mathbb{E} \log T'(X^*) + (1 - \lambda)\mathbb{E} \log U'(Y^*)
\end{align*}
\]

Then subtract

\[
\lambda h(X) + (1 - \lambda)h(Y) = \lambda h(X^*) + (1 - \lambda)h(Y^*) + \lambda \mathbb{E} \log T'(X^*) + (1 - \lambda)\mathbb{E} \log U'(Y^*):
\]

\[
\begin{align*}
    h(\sqrt{\lambda X} + \sqrt{1 - \lambda Y}) - \lambda h(X) - (1 - \lambda)h(Y) \\
    &\geq h(\sqrt{\lambda X^*} + \sqrt{1 - \lambda Y^*}) - \lambda h(X^*) + (1 - \lambda)h(Y^*) = 0
\end{align*}
\]
A Proof that Shannon Missed (Cont’d)

\[ h(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) = h(\sqrt{\lambda}T(X^*) + \sqrt{1 - \lambda}U(Y^*)) \]
\[ = h(\sqrt{\lambda}T(\sqrt{\lambda}\tilde{X} - \sqrt{1 - \lambda}\tilde{Y}) + \sqrt{1 - \lambda}U(\sqrt{1 - \lambda}\tilde{X} + \sqrt{\lambda}\tilde{Y})) \]
\[ \geq h(\sqrt{\lambda}T(\sqrt{\lambda}\tilde{X} - \sqrt{1 - \lambda}\tilde{Y}) + \sqrt{1 - \lambda}U(\sqrt{1 - \lambda}\tilde{X} + \sqrt{\lambda}\tilde{Y})|\tilde{Y}) \]
\[ = h(\tilde{X}|\tilde{Y}) + \mathbb{E} \log(\lambda T'(\sqrt{\lambda}\tilde{X} - \sqrt{1 - \lambda}\tilde{Y}) + (1 - \lambda)U'(\sqrt{1 - \lambda}\tilde{X} + \sqrt{\lambda}\tilde{Y})) \]
\[ = h(\tilde{X}) + \mathbb{E} \log(\lambda T'(X^*) + (1 - \lambda)U'(Y^*)) \]
\[ \geq h(\sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^*) + \lambda \mathbb{E} \log T'(X^*) + (1 - \lambda)\mathbb{E} \log U'(Y^*) \]

Then subtract

\[ \lambda h(X) + (1 - \lambda)h(Y) = \lambda h(X^*) + (1 - \lambda)h(Y^*) + \lambda \mathbb{E} \log T'(X^*) + (1 - \lambda)\mathbb{E} \log U'(Y^*) \]
\[ h(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) - \lambda h(X) - (1 - \lambda)h(Y) \]
\[ \geq h(\sqrt{\lambda}X^* + \sqrt{1 - \lambda}Y^*) - \lambda h(X^*) + (1 - \lambda)h(Y^*) = 0 \]

Equality case: \( T' = U' = \text{Cst} \) hence \( X \sim X^*, Y \sim Y^*. \)
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Conclusion for Rényi’s Entropy

\[ h_r(\sqrt{\lambda X} + \sqrt{1-\lambda Y}) - \lambda h_p(X) - (1 - \lambda)h_q(Y) \geq h_r(\sqrt{\lambda X^*} + \sqrt{1-\lambda Y^*}) - \lambda h_p(X^*) - (1 - \lambda)h_q(Y^*) \]

\[ = \frac{r}{2(r-1)} \left( \frac{\log r}{r} - \frac{\log p}{p} - \frac{\log q}{q} \right) \]

where \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \) (Young’s triple with rate \( \lambda \)), i.e., where Hölder conjugates satisfy \( \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'} \).

Equality case: \( T' = U' = \text{Cst} \) hence \( X \propto X^*, \ Y \propto Y^* \).
Conclusion for Rényi’s Entropy

\[ h_r(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) - \lambda h_p(X) - (1 - \lambda)h_q(Y) \geq h_r(\sqrt{\lambda}X^* + \sqrt{1-\lambda}Y^*) - \lambda h_p(X^*) - (1 - \lambda)h_q(Y^*) \]

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Equality case: \( T' = U' = \text{Cst} \) hence \( X \propto X^* \), \( Y \propto Y^* \).

- the natural generalization of the EPI for Rényi entropies
Conclusion for Rényi’s Entropy

\[
h_r(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) - \lambda h_p(X) - (1 - \lambda)h_q(Y) \\
\geq h_r(\sqrt{\lambda}X^* + \sqrt{1-\lambda}Y^*) - \lambda h_p(X^*) - (1 - \lambda)h_q(Y^*) \\
= \frac{r}{2(r-1)} \left( \frac{\log r}{r} - \frac{\log p}{p} - \frac{\log q}{q} \right)
\]

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Equality case: \( T' = U' = \text{Cst} \) hence \( X \propto X^*, \ Y \propto Y^* \).

- the natural generalization of the EPI for Rényi entropies
- turns out to be equivalent to strong Young’s inequality and its reverse
  [Dembo, Cover, Thomas, 1991] [Barthe, 1998]
Conclusion (Rényi’s Entropy)

For \( N \geq 2 \) variables:

\[
h_r(\sum_i \sqrt{\lambda_i} X_i) - \sum_i \lambda_i h_{r_i}(X_i) \text{ is minimum for } X_i \text{ i.i.d. Gaussian}: \\
h_r(\sum_i \sqrt{\lambda_i} X_i) \geq \sum_i \lambda_i h_{r_i}(X_i) + \frac{1}{2} \left( \log \frac{r}{r'} - \sum_i \log \frac{r_i}{r'_i} \right)
\]

where Hölder conjugates satisfy \( \frac{1}{r'} = \sum_i \frac{1}{r'_i} \) where \( \frac{1}{r_i} = \frac{\lambda_i}{r'} \)
Conclusion (Rényi’s Entropy)

For $N \geq 2$ variables:

$$h_r(\sum_i \sqrt{\lambda_i} X_i) - \sum_i \lambda_i h_r(X_i)$$ is minimum for $X_i$ i.i.d. Gaussian:

$$h_r(\sum_i \sqrt{\lambda_i} X_i) \geq \sum_i \lambda_i h_r(X_i) + \frac{r'}{2} \left( \frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i} \right)$$

where Hölder conjugates satisfy $\frac{1}{r'} = \sum_i \frac{1}{r'_i}$ where $\frac{1}{r'_i} = \frac{\lambda_i}{r_i}$

In particular for $r > 1$, then $r \geq r_i$, $h_r(X) \leq h_{r_i}(X)$

For any for any convex combination ($\sum_i \lambda_i = 1$), choosing $r'_i = r' / \lambda_i$:

$$h_r(\sum_i \sqrt{\lambda_i} X_i) \geq \sum_i \lambda_i h_r(X_i) + \frac{r'}{2} \left( \frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i} \right)$$
Back to the Rényi EPI

Theorem (Rényi Entropy Power Inequality)

\[ e^{2h_r(\sum_i X_i)} \geq c \cdot \sum_i e^{2h_r(X_i)} \]

\[ \iff \] for any convex combination (\( \sum_i \lambda_i = 1 \))

\[ h_r(\sum_i \sqrt{\lambda_i}X_i) \geq \sum_i \lambda_i h_r(X_i) + \log \sqrt{c} \]
Back to the Rényi EPI

Theorem (Rényi Entropy Power Inequality)

\[
e^{2h_r(\sum_i X_i)} \geq c \cdot \sum_i e^{2h_r(X_i)}
\]

\[\iff \text{for any convex combination } (\sum_i \lambda_i = 1)\]

\[
h_r\left(\sum_i \sqrt{\lambda_i} X_i\right) \geq \sum_i \lambda_i h_r(X_i) + \log \sqrt{c}
\]

We have found (for \(r > 1\)):

\[
\log \sqrt{c} = \min\left\{\frac{r'}{2} \left(\frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i}\right) \mid \text{s.t. } \sum_i \frac{1}{r_i} = n - \frac{1}{r'}\right\}
\]
Optimal Constant

$$\log c = \min \left\{ r' \left( \frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i} \right) \right\}$$

s.t. \( \sum_i \frac{1}{r_i} = n - \frac{1}{r'} \)
Optimal Constant

\[ \log c = \min \left\{ r' \left( \frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i} \right) \text{ s.t. } \sum_i \frac{1}{r_i} = n - \frac{1}{r'} \right\} \]

But by the log-sum inequality:

\[ \sum_i x_i \log \frac{x_i}{y_i} \geq \sum_i x_i \log \frac{\sum_i x_i}{\sum_i y_i} \text{ with equality iff } x_i \propto y_i \]
Optimal Constant

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\[ \sum_i \frac{\log r_i}{r_i} = -\sum_i \frac{1}{r_i} \log \frac{1}{r_i} \leq -\sum_i \frac{1}{r_i} \log \frac{\sum_i \frac{1}{r_i}}{N} = -(N - \frac{1}{r'}) \log \frac{N - \frac{1}{r'}}{N} \]

with equality iff \( r_i \) are equal.
Optimal Constant

\[
\log c = \min \left\{ \frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i} \right\} \text{ s.t. } \sum_i \frac{1}{r_i} = n - \frac{1}{r'}
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\]

with equality iff \( r_i \) are equal.

This gives

\[
\log c = \frac{\log r}{r - 1} + (Nr' - 1) \log \left( 1 - \frac{1}{N} \right)
\]
Optimal Constant

\[ \log c = \min \left\{ r' \left( \frac{\log r}{r} - \sum_i \frac{\log r_i}{r_i} \right) \right\} \text{ s.t. } \sum_i \frac{1}{r_i} = n - \frac{1}{r'} \]

But by the log-sum inequality:

\[ \sum_i x_i \log \frac{x_i}{y_i} \geq \sum_i x_i \log \frac{\sum_i x_i}{\sum_i y_i} \quad \text{with equality iff } x_i \propto y_i \]

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with equality iff \( r_i \) are equal.

This gives

\[ c = r \frac{1}{r-1} \left( 1 - \frac{1}{N r'} \right)^{N r' - 1} \]

which was found by [Ram&Sason,2016] as an improvement of [Bobkov&Chistyakov,2015] (for which \( c = r \frac{1}{r-1} / e \))
Thank you!
Questions?